Turbulent Shear Flows

In the previous chapter on transition, we observed that transition zone is a buffer zone that separates orderly, steady (some times) 2-D laminar flows from highly unsteady, 2-D turbulent flows which contain fluctuating components of velocity & pressure u', v', w' and $p'(x,y,z,t)$. Turbulent flows exist over airfoils, wings & aircraft, within channels or tubes, in jets and in the wake behind bluff geometries. It is the purpose of this chapter to discuss some of the basic characteristics of such turbulent flows, and to discuss numerical modeling of turbulent flows, occuring over the surface of airfoils.

We restrict our study to 2-D turbulent flows. Such flows are characterized by 3-D velocities $u(x,y,z,t)$, $v(x,y,z,t)$, $w(x,y,z,t)$ and pressure $p(x,y,z,t)$ which may be made up of 2-D mean flow components $\overline{U}(x,y)$, $\overline{V}(x,y)$ and mean pressure $\overline{P}(x,y)$ and fluctuating components $u'(x,y,z,t)$, $v'(x,y,z,t)$, $w'(x,y,z,t)$ and $p'(x,y,z,t)$. That is

$$
\begin{aligned}
u(x,y,z,t) &= \overline{U}(x,y) + u'(x,y,z,t) \\
v(x,y,z,t) &= \overline{V}(x,y) + v'(x,y,z,t) \\
w(x,y,z,t) &= w'(x,y,z,t) \\
p(x,y,z,t) &= \overline{P}(x,y) + p'(x,y,z,t)
\end{aligned}
\qquad (1)
$$

The mean flow components are simply averages of the unsteady flow components over a sufficiently large time period T. That is,

I called these → $U = \bar{u}(x,y) = \frac{1}{T}\int_0^T u(x,y,z,t)\,dt$

$V = \bar{v}(x,y) = \frac{1}{T}\int_0^T v(x,y,z,t)\,dt$

$P = \bar{p}(x,y) = \frac{1}{T}\int_0^T p(x,y,z,t)\,dt$ ②

From equation set ① & ②, it follows that

$$\frac{1}{T}\int u'\,dt = \frac{1}{T}\int v'\,dt = \frac{1}{T}\int w'\,dt \equiv 0$$

That is, the fluctuation component averages out to zero if the time average is done over a sufficiently long period of time.

The time averages of other quantities such as $u^2(x,y,z,t)$ may be similarly be defined.

$$\overline{u^2} = \frac{1}{T}\int_0^T u^2(x,y,z,t)\,dt$$

$$= \frac{1}{T}\int_0^T \left[\bar{U}(x,y) + u'(x,y,z,t)\right]^2 dt$$

$$= \frac{1}{T}\int \left[\bar{U}^2 + 2\bar{U}u' + u'^2\right] dt$$

$$= \bar{U}^2 + \frac{1}{T}\int u'^2\,dt = \bar{U}^2 + \overline{u'^2}$$

Reynolds Averaged Navier-Stokes equations: The intantaneous velocity field $u(x,y,t)$, $v(x,y,t)$ and $w(x,y,z,t)$ satisfy the 3-D unsteady Navier-Stokes eqns, plus continuity:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Using
$$u = U(x,y) + u' = U + u'$$
$$v = V + v'$$
$$w = W + w' = w' \quad (\text{for 2-D flows}),$$

$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} + \frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} = 0.$$

If we now take time average of the above equation term by term, then,

$$\overline{\frac{\partial U}{\partial x}} = \frac{1}{T}\int \frac{\partial U}{\partial x}\,dt = \frac{\partial U}{\partial x}$$

$$\overline{\frac{\partial V}{\partial y}} = \frac{1}{T}\int \frac{\partial U}{\partial x}\,dt = \frac{\partial V}{\partial y}$$

$$\overline{\frac{\partial u'}{\partial x}} = \frac{1}{T}\int_0^T \frac{\partial}{\partial x} u'\,dt = \frac{1}{T}\frac{\partial}{\partial x}\int_0^T u'\,dt = 0$$

Likewise, $\overline{\frac{\partial v'}{\partial y}} = 0$; $\overline{\frac{\partial w'}{\partial z}} = 0$

Thus, the Reynolds averaged continuity eqn for 2-D flows is

$$\boxed{\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0}$$

The u-momentum eqn is

$$\rho\left[\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}(u^2) + \frac{\partial}{\partial y}(uv) + \frac{\partial}{\partial z}(uw)\right] + \frac{\partial p}{\partial x} = \nu \nabla^2 u$$

Taking time average term by term,

$$\overline{\frac{\partial u}{\partial t}} = \frac{\partial U}{\partial t} + \frac{1}{T}\int_0^T \frac{\partial u'}{\partial t}\,dt = \frac{\partial U}{\partial t} = 0 \quad \text{since } U \neq f(t)$$

Likewise, using relations derived earlier,

$$\overline{\frac{\partial}{\partial x} u^2} = \frac{\partial}{\partial x}(U^2) + \frac{\partial}{\partial x}\left(\overline{u'^2}\right)$$

$$\overline{\frac{\partial (uv)}{\partial y}} = \frac{\partial}{\partial y}(UV) + \frac{\partial}{\partial y}\left(\overline{u'v'}\right)$$

$$\overline{\frac{\partial}{\partial z}(uw)} = \frac{\partial}{\partial z}(UW) + \frac{\partial}{\partial z}\left(\overline{u'w'}\right) = \frac{\partial}{\partial z}\left(\overline{u'w'}\right) \quad \text{since } W = 0$$

$$= 0 \quad \text{since } \frac{\partial}{\partial z}(\ ) \equiv 0$$

Flow is 2-D.

$$\overline{\frac{\partial p}{\partial x}} = \frac{\partial P}{\partial x}$$

$$\nu\overline{\nabla^2 u} = \nu \nabla^2 U + \nu\overline{\nabla^2 u'} = \nu \nabla^2 U$$

Thus, the u-momentum equation becomes:

$$\frac{\partial}{\partial x}(U^2) + \frac{\partial}{\partial y}(UV) + \frac{1}{\rho}\frac{\partial}{\partial x} p = \nu \nabla^2 U - \frac{\partial}{\partial x}\left(\overline{u'^2}\right) - \frac{\partial}{\partial y}\left(\overline{u'v'}\right)$$

If we use continuity, then, u-momentum equation becomes

$$\boxed{U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y} + \frac{1}{\rho}\frac{\partial P}{\partial x} = \nu \nabla^2 U - \left(\overline{u'^2}\right)_x - \left(\overline{u'v'}\right)_y}$$

The v-momentum equation likewise is

$$\boxed{U\frac{\partial V}{\partial x} + V\frac{\partial V}{\partial y} + \frac{1}{\rho}\frac{\partial p}{\partial y} = \nu\,\nabla^2 V - (\overline{u'v'})_x - (\overline{v'^2})_y}$$

These three equations collectively are called the Reynolds averaged Navier-Stokes equations.

In thin shear layers(wall-bound) or free, these equations may be approximated, as Prandtl did originally with the laminar flow equations. The result is

$$\boxed{\begin{gathered} U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y} + \frac{1}{\rho}\frac{\partial P}{\partial x} = \nu\frac{\partial^2 U}{\partial y^2} - \frac{\partial}{\partial y}(\overline{u'v'}) \\ \frac{\partial P}{\partial y} = 0 \\ \frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0 \end{gathered}}$$