Derivation of Compressible Potential Flow Equation Based on Chapter 11 of Anderson's Aerodynamics

Lakshmi N Sankar
lsankar@ae.gatech.edu

Motivation

- Between 1903 (Wright Brothers’ first flight) and 1940s, aircraft technology rapidly grew in a number of fronts – aerodynamics, structures, propulsion, control.
- The two world wars (1914-1918 and 1939-1945), created needs and the spurred the development of faster aircraft, from low subsonic speeds to higher speeds – subsonic, transonic, and supersonic speeds.
- Civilian aircraft benefited from these developments.

Motivation (Continued)

- Prandtl is credited with pointing the similarity between incompressible flow equations and compressible potential flow equation during the late 1920s, and presented his work in a seminar in 1922.
- Glauert, 1928, independently developed/published 2-D rules and these came to be known as Prandtl-Glauert rule.
- In 1946, Gothert extended and modified these rules to 3-D flows.
- In 1928, Ackeret developed equations for supersonic flow.

In the United States..

- US lagged behind Europe in advancements.
- NACA was formed in 1915 to reverse this trend.
  - Initially advised government on policy matters (air mail, patent issues, pilot licensing,…)
  - http://www.centennialofflight.gov/essay/Evolution_of_Technology/NACA/Tech1.htm
- Langley Labs (~1925) started looking at airfoils, wings, drag reduction in their wind tunnels.
- See the above link for more on US activities.

Preliminary Remarks

- In our low speed aerodynamics (Chapter 2 in text), we combined continuity equation (divergence of velocity is zero), and the irrotational flow assumption (curl of velocity is zero, velocity is gradient of velocity potential) to get Laplace's equation for ϕ .
- In this lecture, we discuss how the governing equations become different for compressible flow.

Assumptions

- Flow is steady
- Inviscid (thin boundary layer regime is handled separately), only the outer inviscid flow away from wall boundary layers is considered.
- Irrotational
- Reversible (no shocks)
- Adiabatic (no heat addition)
- We will do the derivations in 2-D for brevity, and write down the 3-D equivalent expressions.

Reversible, adiabatic flow is isentropic

$$p = C\rho^{\gamma}$$

$$dp = \gamma C\rho^{\gamma-1}d\rho$$

$$Or,$$

$$dp = \gamma \frac{p}{\rho} d\rho$$

$$= \gamma RT d\rho$$

$$= a^{2} d\rho$$

a : Speed of sound

Continuity Equation

$$\frac{\partial \rho}{\partial t} + \nabla \bullet \left(\rho \vec{V}\right) = 0$$

For steady, 2-D flow, we get:

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho \mathrm{v})}{\partial y} = 0$$

Use product rule:

$$\rho \frac{\partial u}{\partial x} + u \frac{\partial \rho}{\partial x} + \rho \frac{\partial \mathrm{v}}{\partial y} + \mathrm{v} \frac{\partial \rho}{\partial y} = 0$$

Rearrange:

$$\rho \left[\frac{\partial u}{\partial x} + \frac{\partial \mathrm{v}}{\partial y} \right] + u \frac{\partial \rho}{\partial x} + \mathrm{v} \frac{\partial \rho}{\partial y} = 0$$

Continuity (continued)

From the previous slide:

$$\rho\left[\frac{\partial u}{\partial x}+\frac{\partial \mathrm{v}}{\partial y}\right]+u\frac{\partial \rho}{\partial x}+\mathrm{v}\frac{\partial \rho}{\partial y}=0$$

Isentropic Gas relation says:

$$dp = a^2 d\rho$$

Thus,

$$\frac{\partial \rho}{\partial \mathrm{x}}=\frac{1}{a^2}\frac{\partial p}{\partial x}$$

and,

$$\frac{\partial \rho}{\partial \mathrm{y}}=\frac{1}{a^2}\frac{\partial p}{\partial y}$$

Continuity (Continued)

Rearranging the equations from the previous slide, replacing density dervatives with pressure derivatives, we get

$$\rho\left[\frac{\partial u}{\partial x}+\frac{\partial \mathrm{v}}{\partial y}\right]+\frac{1}{a^2}\left[u\frac{\partial p}{\partial x}+\mathrm{v}\frac{\partial p}{\partial y}\right]=0$$

We next invoke Euler equation: dp =- ρVdV

$$dp=-\rho d\left[\frac{V^2}{2}\right]=-\frac{1}{2}\rho d\left[u^2+\mathrm{v}^2\right]$$

$$=-\rho u du-\rho \mathrm{v} d\mathrm{v}$$

Thus,

$$\frac{\partial p}{\partial x}=-\rho u\frac{\partial u}{\partial x}-\rho \mathrm{v}\frac{\partial \mathrm{v}}{\partial x}$$

$$\frac{\partial p}{\partial y}=-\rho u\frac{\partial u}{\partial y}-\rho \mathrm{v}\frac{\partial \mathrm{v}}{\partial y}$$

Continuity (Concluded)

Replacing the pressure derivatives at the top of the previous slide, with those Found at the bottom of the previous slide, we get, after some light algebra,

$$\left(1-\frac{u^2}{a^2}\right)\frac{\partial u}{\partial x}-\left(2\frac{\mathrm{uv}}{a^2}\right)\frac{\partial u}{\partial y}+\left(1-\frac{\mathrm{v}^2}{a^2}\right)\frac{\partial \mathrm{v}}{\partial y}=0$$

We have used irrotationality : $\frac{\partial u}{\partial y}-\frac{\partial \mathrm{v}}{\partial x}=0$ to replace ∂v/ ∂x

Finally, invoke potential flow assumption: Velocity is gradient of velocity potential

$$u=\frac{\partial \phi}{\partial x}$$

$$\mathrm{v}=\frac{\partial \phi}{\partial y}$$

Compressible Potential Flow Equation

- We get:

$$\left(1-\frac{u^2}{a^2}\right)\frac{\partial^2\phi}{\partial x^2}-\left(2\frac{\mathrm{uv}}{a^2}\right)\frac{\partial^2\phi}{\partial x\partial y}+\left(1-\frac{\mathrm{v}^2}{a^2}\right)\frac{\partial^2\phi}{\partial y^2}=0$$

Observations:

It is a second order PDE
It is highly non-linear. Superposition of sources, sinks, vortices will not work, unless we linearize it
The non-linear PDE reduces to Laplace’s equation in incompressible flow, since speed of sound approaches infinity relative to flow velocity.

In 3-D we get a similar equation

$$\left(1-\frac{u^2}{a^2}\right)\frac{\partial^2\phi}{\partial x^2}-\left(2\frac{\mathrm{uv}}{a^2}\right)\frac{\partial^2\phi}{\partial x\partial y}+\left(1-\frac{\mathrm{v}^2}{a^2}\right)\frac{\partial^2\phi}{\partial y^2}$$

$$+\left(1-\frac{w^2}{a^2}\right)\frac{\partial^2\phi}{\partial \mathrm{z}^2}-\left(2\frac{\mathrm{uw}}{a^2}\right)\frac{\partial^2\phi}{\partial x\partial \mathrm{z}}-\left(2\frac{\mathrm{vw}}{a^2}\right)\frac{\partial^2\phi}{\partial y\partial \mathrm{z}}=0$$

Compressible Potential Flow Equation

- We derived:

$$\left(1-\frac{u^2}{a^2}\right)\frac{\partial^2\phi}{\partial x^2}-\left(2\frac{\mathrm{uv}}{a^2}\right)\frac{\partial^2\phi}{\partial x\partial y}+\left(1-\frac{\mathrm{v}^2}{a^2}\right)\frac{\partial^2\phi}{\partial y^2}=0$$

$Or,$

$$\left(a^2-u^2\right)\frac{\partial^2\phi}{\partial x^2}-\left(2\mathrm{uv}\right)\frac{\partial^2\phi}{\partial x\partial y}+\left(a^2-\mathrm{v}^2\right)\frac{\partial^2\phi}{\partial y^2}=0$$

Observations:

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Small Disturbance Assumptions

- We want to linearize this equation, to get an equation that resembles Laplace’s equation for subsonic flows.
- This is possible under some restrictive assumptions.
  - We have already assumed steady, inviscid, irrotational, isentropic flow to derive the equation in the previous slide (see lecture #3)
  - We deal with subsonic flow or supersonic flow. We do not deal with transonic flow with mixed subsonic and supersonic regimes.
  - The bodies (airfoils, wings, bodies) were analyzing are thin (low thickness to chord ratio), have mild or small camber (a few percentage), and operate at low angles of attack.
  - As a result, the freestream is disturbed only very slightly by the presence of the body (airfoil, wing, or body of revolution) placed in the stream.

Disturbance Potential, φ

- We introduce a disturbance potential, φ, related to the full potential ϕ as follows:

$$\phi = V_\infty x + \varphi$$

It must be noted that in this course we are using the wind tunnel coordinate system, and the freestream velocity is parallel to the x- axis.

From the above equation, we obtain the flow velocity components as follows.
Note: subscripts x,y represent derivatives with respect to x and y

$$u = \phi_x = V_\infty + \varphi_x$$

$$v = \phi_y = \varphi_y$$

From the small disturbance assumptions, it then follows that the disturbance velocities are smaller than freestream.

$$\frac{\varphi_x}{V_\infty} \langle\langle 1$$

$$\frac{\varphi_y}{V_\infty} \langle\langle 1$$

Derivation of Linearized Potential Flow Equations (Continued)

From previous slide, $\phi = V_\infty x + \varphi$

Take second derivatives:

$$\phi_{xx} = \varphi_{xx}; \phi_{yy} = \varphi_{yy}; \phi_{xy} = \varphi_{xy}$$

Next, consider the second term in the nonlinear potential flow equation, $-2uv\phi_{xy}$. This term may be viewed as

$$\begin{aligned} 2uv\varphi_{xy} &= 2\left(V_\infty + \varphi_x\right)\varphi_y \varphi_{xy} \\ &= V_\infty^2 \left(1 + \frac{\varphi_x}{V_\infty}\right)\left(\frac{\varphi_y^2}{V_\infty^2}\right)_x \cong 0 \end{aligned}$$

Small since it is a second order term!!

Energy Equation

- Energy equation from lecture #1 may be written as:

$$C_pT + \frac{V^2}{2} = C_pT_\infty + \frac{V_\infty^2}{2}$$

From thermodynamics,

$$C_p - C_\text{v} = R$$

$$\text{Use } \frac{\mathrm{C_p}}{\mathrm{C_v}} = \gamma$$

$$\text{We get}: \mathrm{C_p} = \frac{\gamma}{(\gamma - 1)}R$$

Recall $\gamma RT=a^2$

The energy equation becomes

$$\frac{a^2}{\gamma - 1} + \frac{u^2 + v^2}{2} = \frac{a_\infty^2}{\gamma - 1} + \frac{V_\infty^2}{2}$$

Exercise

- Show that the energy equation

$$C_p T + \frac{V^2}{2} = C_p T_\infty + \frac{V_\infty^2}{2}$$

Can be manipulated to yield

$$\frac{T}{T_\infty} = 1 + \frac{(\gamma - 1)}{2} M_\infty^2 \left\{ 1 - \frac{u^2 + \mathrm{v}^2}{V_\infty^2} \right\}$$

Derivation of Linearized Potential Flow Equations (Continued)

Next, consider the coefficient a^2-u^2. This appears in our nonlinear potential flow Equation.

We can linearize this coefficient as follows, using energy equation at the bottom of the previous slide:

$$
\begin{aligned}
a^2 - u^2 &= a_\infty^2 + \frac{\gamma - 1}{2}\left[V_\infty^2 - u^2 - v^2\right] - u^2 \\
&= a_\infty^2 + \frac{\gamma - 1}{2} V_\infty^2 \left[1 - \left(1 + \frac{\varphi_x}{V_\infty}\right)^2 - \left(\frac{\varphi_y}{V_\infty}\right)^2\right] - V_\infty^2 \left(1 + \frac{\varphi_x}{V_\infty}\right)^2 \\
&\cong a_\infty^2 - V_\infty^2 - (\gamma + 1) V_\infty^2 \left(\frac{\varphi_x}{V_\infty}\right) \\
&= a_\infty^2 \left[1 - M_\infty^2 - (\gamma + 1) M_\infty^2 \left(\frac{\varphi_x}{V_\infty}\right)\right]
\end{aligned}
$$

Derivation of Linearized Potential Flow Equations (Continued)

$$\begin{aligned}
a^2 - u^2 &= a_\infty^2 + \frac{\gamma - 1}{2}\left[V_\infty^2 - u^2 - v^2\right] - u^2 \\
&= a_\infty^2 + \frac{\gamma - 1}{2} V_\infty^2 \left[1 - \left(1 + \frac{\varphi_x}{V_\infty}\right)^2 - \left(\frac{\varphi_y}{V_\infty}\right)^2\right] - V_\infty^2 \left(1 + \frac{\varphi_x}{V_\infty}\right)^2 \\
&\cong a_\infty^2 - V_\infty^2 - (\gamma + 1) V_\infty^2 \left(\frac{\varphi_x}{V_\infty}\right) \\
&= a_\infty^2 \left[1 - M_\infty^2 - (\gamma + 1) M_\infty^2 \left(\frac{\varphi_x}{V_\infty}\right)\right]
\end{aligned}$$

Examine the above approximation carefully. Note that we have neglected second powers of the "disturbance velocities divided by freestream" as small.

We have, however, kept the first power of the term φ_x/V_∞ .

This is because the term $(1-M_\infty^2)$ itself may be small in transonic flows.

Derivation of Linearized Potential Flow Equations (Continued)

In a very similar manner, we can show that

$$a^2 - v^2 \cong a_\infty^2$$

Exercise: Verify this.

With these approximations, the quasi-linear form of the full potential equation takes on the following simpler, still nonlinear, form for transonic flows:

$$\left[1 - M_\infty^2 - (\gamma + 1) M_\infty^2 \varphi_x\right] \varphi_{xx} + \varphi_{yy} = 0$$

In subsonic and supersonic flows, the small disturbance equation becomes

$$\left(1 - \mathrm{M}_\infty^2\right) \varphi_{xx} + \varphi_{yy} = 0$$

Pressure Coefficient

Recall the definition of the pressure coefficient as the ratio,
Gauge pressure/freestream dynamic pressure

$$C_p = \frac{p - p_\infty}{\frac{1}{2}\rho V_\infty^2}$$

Divide numerator and denominator by p_∞ , and use equation of state

$p_\infty = \rho R T_\infty$

and the definition of speed of sound : $a_\infty^2 = \gamma R T_\infty$

We get: $$C_p = \frac{\frac{p}{p_\infty} - 1}{\frac{\gamma}{2} M_\infty^2}$$

Pressure Coefficient (Continued)

In isentropic flows, pressure and temperature are related by

$$\frac{p}{p_\infty} = \left(\frac{T}{T_\infty}\right)^{\frac{\gamma}{\gamma-1}}$$

The equation on the previous slide becomes

$$C_p = \frac{\frac{p}{p_\infty} - 1}{\frac{\gamma}{2} M_\infty^2} = \frac{\left(\frac{T}{T_\infty}\right)^{\frac{\gamma}{\gamma-1}} - 1}{\frac{\gamma}{2} M_\infty^2}$$

Pressure Coefficient (Continued)

- Using an exercise assigned earlier, we get

$$C_p = \frac{\dfrac{p}{p_\infty} - 1}{\dfrac{\gamma}{2} M_\infty^2} = \frac{\left(\dfrac{T}{T_\infty}\right)^{\frac{\gamma}{\gamma-1}} - 1}{\dfrac{\gamma}{2} M_\infty^2} = \frac{\left\{1 + \dfrac{\gamma - 1}{2} M_\infty^2 \left(1 - \dfrac{u^2 + \mathrm{v}^2}{V_\infty^2}\right)\right\}^{\frac{\gamma}{\gamma-1}} - 1}{\dfrac{\gamma}{2} M_\infty^2}$$

Linearization for Small disturbance Flows

$$C_p = \frac{\left[1+\frac{\gamma-1}{2}M_\infty^2\left(1-\frac{u^2+v^2}{V_\infty^2}\right)\right]^{\frac{\gamma}{\gamma-1}}-1}{\frac{\gamma}{2}M_\infty^2}$$

$$= \frac{\left[1+\frac{\gamma-1}{2}M_\infty^2\left(1-\frac{(V_\infty+\varphi_x)^2+\varphi_y^2}{V_\infty^2}\right)\right]^{\frac{\gamma}{\gamma-1}}-1}{\frac{\gamma}{2}M_\infty^2}$$

$$\approx \frac{\left[1+\frac{\gamma-1}{2}M_\infty^2\left(-2\frac{\varphi_x}{V_\infty}\right)\right]^{\frac{\gamma}{\gamma-1}}-1}{\frac{\gamma}{2}M_\infty^2}$$

In arriving at the above form, we have neglected second powers of terms such as φ_x/V_∞ and φ_y/V_∞ .

Linearization for Small disturbance Flows (Continued)

Next, we use the binomial expansion

$$(1+\varepsilon)^n \approx 1+n\varepsilon$$

where ε is any small quantity ε << 1 and n is any real number. Then,

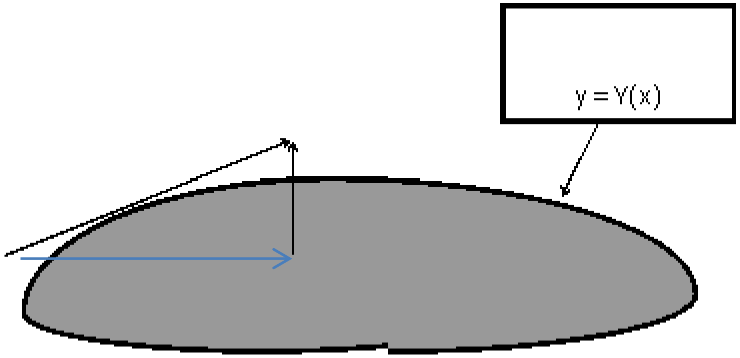
$$C_p \approx \frac{\left[1-\frac{\gamma}{2}M_\infty^2\varphi_x-1\right]}{\frac{\gamma}{2}M_\infty^2}$$

Simplifying, the following simple, small disturbance, approximation to surface pressure coefficient results

$$C_p \approx -\frac{2\varphi_x}{V_\infty}$$

$\vec{V}$

Boundary Conditions at the Body Surface



At any point on the body surface, the flow must be tangential to the body.

In other words, the slope of the velocity vector must equal the body slope.

$$\frac{v}{u} = \frac{\varphi_y}{V_\infty + \varphi_x} = \frac{dY}{dx}$$

Or,

$$\varphi_y = \left(V_\infty + \varphi_x\right)\frac{dY}{dx} = V_\infty\left(1 + \frac{\varphi_x}{V_\infty}\right)\frac{dY}{dx}$$

Linearized Boundary Conditions at the Body Surface

Exact:

$$\frac{v}{u} = \frac{\varphi_y}{V_\infty + \varphi_x} = \frac{dY}{dx}$$

$Or,$

$$\varphi_y = (V_\infty + \varphi_x)\frac{dY}{dx} = V_\infty\left(1 + \frac{\varphi_x}{V_\infty}\right)\frac{dY}{dx}$$

small

Also small for thin body
Small camber, small alpha

Linearized:

$$\varphi_y \approx V_\infty \frac{dY}{dx}$$

Summary

- Linearized Governing equation
- Linearized expression for Pressure Coefficient
- Linearized Boundary Condition

$$\left(1 - \mathrm{M}_\infty^2\right)\varphi_{xx} + \varphi_{yy} = 0$$

$$C_p \approx -\frac{2\varphi_x}{V_\infty}$$

$$\varphi_y \approx V_\infty \frac{dY}{dx}$$