

AE 3330: Lecture 12  
Prandtl-Glauert Rule for 2-D Subsonic Flow

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# Linearized Compressible Potential Flow

Recall the governing equation:

$$\beta^2 \varphi_{xx} + \varphi_{yy} = 0$$

where,

$$\beta = \sqrt{1 - M_\infty^2}$$

The boundary condition requiring the flow be tangential to the airfoil surface  $y = Y(x)$  given in the wind tunnel coordinate system, where the freestream is parallel to the x-axis.

$$\varphi_y = V_\infty \frac{dY}{dx}$$

Resulting surface pressure coefficient is  $C_p = -2 \frac{\varphi_x}{V_\infty}$

# Linearized Potential Flow (Continued)

The result we are most interested in is the surface pressure coefficient  $C_p$ .

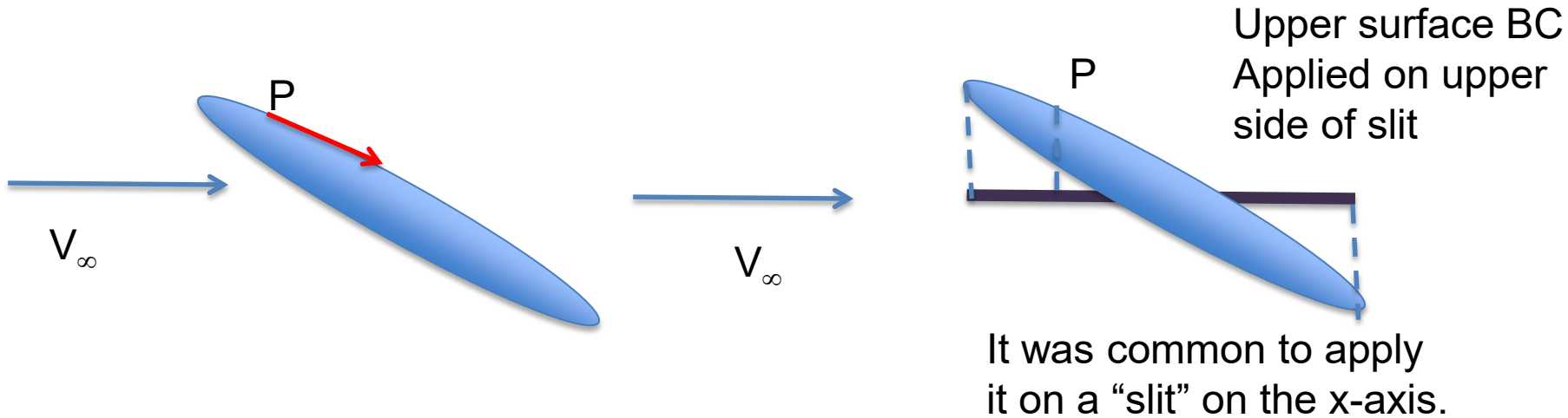
If we know  $C_p$ , we can integrate it to get  $C_l$  and  $C_m$

$$C_p = -2 \frac{\varphi_x}{V_\infty}$$

# Where should we apply the tangency Boundary Condition?

- The proper, rigorous, approach would be to directly apply the boundary condition at the actual airfoil surface  $y=Y(x)$
- At the time Prandtl-Glauert rule was being developed, thick panel methods were not computationally feasible. It was common to apply the boundary condition on the x-axis, on a slit that is a projection of the chord line to the x-axis.
- The upper surface tangency condition was applied on the upper side, and the lower surface tangency was applied on the lower side of the slit.

# Where should we apply the boundary condition? Continued..



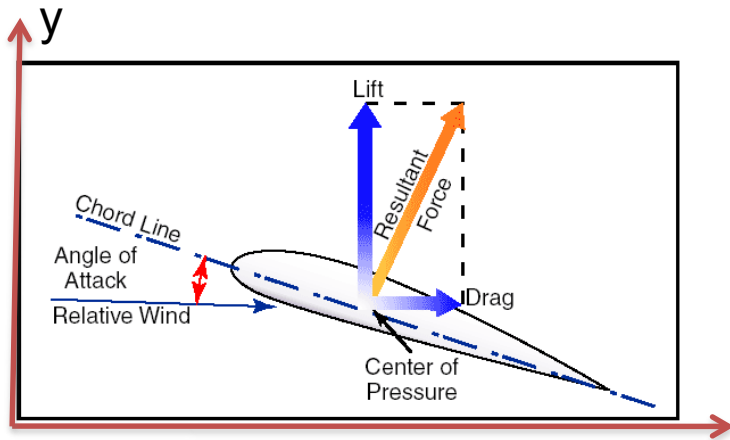
Instead of applying the boundary Condition on  $y=Y(x)$

$$\varphi_y \Big|_{y=Y(x)} = V_\infty \frac{dY}{dx}$$

$$\varphi_y \Big|_{y \approx 0} = V_\infty \frac{dY}{dx}$$

The upper surface tangency condition was applied on the upper side, and the lower surface tangency was applied on the lower side of the slit.

# Compressible Flow vs Incompressible

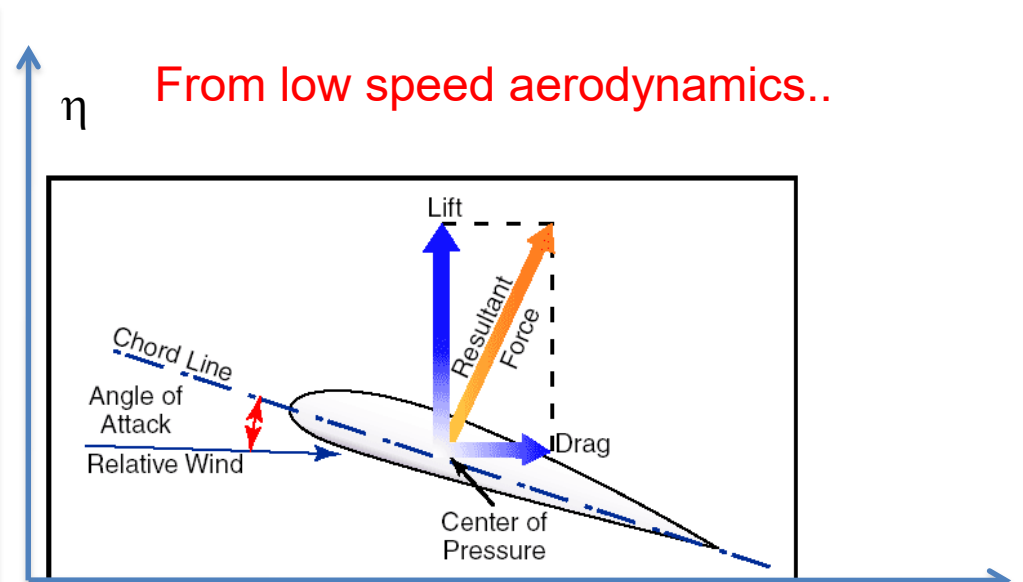


$$\beta^2 \varphi_{xx} + \varphi_{yy} = 0 \quad x$$

where,

$$\beta = \sqrt{1 - M_\infty^2}$$

$$\varphi_y \Big|_{y=0} = V_\infty \frac{dY}{dx} \quad C_p = -2 \frac{\varphi_x}{V_\infty}$$



$$\Phi_{\xi\xi} + \Phi_{\eta\eta} = 0$$

$$\Phi_\eta \Big|_{\eta=0} = V_\infty \frac{dY_1}{d\xi} \quad C_{p1} = -2 \frac{\Phi_\xi}{V_\infty}$$

How do we go from here (compressible)

to there (incompressible)?

# Transformation of Compressible Flow Problem into an Incompressible Flow problem: Why?

- Incompressible flows may be inexpensively modeled using panel methods on personal computers. A number of panel codes written in BASIC, Pascal, C or MATLAB are available.
  - e.g. XFOIL, Panel.m in class notes
- New airfoils are often tested in wind tunnels.
  - It is always easier and less expensive to study or test an airfoil under low speed incompressible flow conditions than under compressible flow conditions.

# Transformation

- We know that our governing equation and boundary condition are linear.
- Therefore, Prandtl and Glauert sought simple linear transformations that will transform the flow from a compressible flow coordinate system  $(x, y)$  to an incompressible flow coordinate system  $(\xi, \eta)$ .
- They assumed the following transformation for the  $x$  and  $y$  axes. Other choices are possible, but this is what they both chose.

$$\begin{aligned}\xi &= x \\ \eta &= \beta y\end{aligned}$$

# Transformation (Continued)

- The disturbance velocity potential  $\Phi$  and the freestream velocity in the incompressible flow regime may be different from the compressible flow regime.
- Prandtl and Glauert assumed that these two are linearly related.

$$\Phi = A\varphi$$

$$V_{\infty 1} = V_{\infty}$$

# Transformation (Continued)

- The airfoil shapes in the compressible flow is  $Y(x)$  and incompressible flow problem is  $Y_1(x)$ . *We assume that they are affinely related.*
- That is, their shapes are linearly related (stretched or compressed by a linear factor) so that their slopes differ from each other only by a constant,  $D$ :

$$\frac{dY}{dx} = D \frac{dY_1}{d\xi}$$

Prandtl and Glauert chose  $D = 1$ . This means same airfoil in both problems

$$\frac{dY}{dx} = \frac{dY_1}{d\xi}$$

# Linear Transformation (summary)

- We have introduced just one constant  $A$  that links the compressible flow properties ( $x$ ,  $y$ ,  $\varphi$ ,  $dY/dx$ , and  $V_\infty$ ) to their counterparts in the incompressible flow problem ( $\xi$ ,  $\eta$ ,  $\Phi$ ,  $dY_1/d\xi$ , and  $V_{\infty 1}$ ).
- Next, we get to work, transforming the equations from one coordinate system to the other.

$$\xi = x \quad \eta = \beta y \quad \Phi = A \varphi \quad dY/dx = dY_1/d\xi \quad V_\infty = V_{\infty 1}$$

We first transform the first and second derivatives of the velocity potential

$$\xi=x \quad \eta=\beta y \quad \Phi = A \varphi \quad dY/dx=dY1/dx \quad V_{\infty} = V_{\infty 1}$$

$$\frac{\partial \varphi}{\partial x} = \frac{\partial}{\partial x} \left( \frac{1}{A} \Phi \right) = \frac{1}{A} \frac{\partial \Phi}{\partial x} = \frac{1}{A} \frac{\partial \Phi}{\partial \xi} \cdot \frac{d\xi}{dx} = \frac{1}{A} \frac{\partial \Phi}{\partial \xi}$$

$$\frac{\partial^2 \varphi}{\partial x^2} = \frac{\partial}{\partial x} \left( \frac{\partial \varphi}{\partial x} \right) = \frac{\partial}{\partial x} \left( \frac{1}{A} \frac{\partial \Phi}{\partial \xi} \right) = \frac{1}{A} \frac{\partial^2 \Phi}{\partial \xi^2} \cdot \frac{d\xi}{dx} = \frac{1}{A} \frac{\partial^2 \Phi}{\partial \xi^2}$$

*Likewise, (Check)*

$$\frac{\partial \varphi}{\partial y} = \frac{\beta}{A} \frac{\partial \Phi}{\partial \eta}$$

$$\frac{\partial^2 \varphi}{\partial y^2} = \frac{\beta^2}{A} \frac{\partial^2 \Phi}{\partial \eta^2}$$

# Inserting the second derivatives into linearized potential flow equation

$$\beta^2 \varphi_{xx} + \varphi_{yy} = 0$$

where,

$$\beta = \sqrt{1 - M_\infty^2}$$

becomes

$$\frac{\beta^2}{A} \frac{\partial^2 \Phi}{\partial \xi^2} + \frac{\beta^2}{A} \frac{\partial^2 \Phi}{\partial \eta^2} = 0$$

We get Laplace's equation  
In the incompressible flow.

$$\Phi_{\xi\xi} + \Phi_{\eta\eta} = 0$$

One down (potential flow equation), two more to go (boundary condition  
And surface pressure coefficient).

# We next work on the boundary condition

$$\xi=x \quad \eta=\beta y \quad \Phi = A\phi \quad dY/dx=dY_1/dx \quad V_\infty = V_{\infty 1}$$

Boundary condition for our compressible flow problem is

$$\phi_y \Big|_{y=0} = V_\infty \frac{dY}{dx}$$

We saw a couple of slides ago  $\frac{\partial \phi}{\partial y} = \frac{\beta}{A} \frac{\partial \Phi}{\partial \eta}$

We get  $\frac{\partial \phi}{\partial y} = \frac{\beta}{A} \frac{\partial \Phi}{\partial \eta} = V_\infty \frac{dY}{dx} = V_{\infty 1} \frac{dY}{dx} = V_{\infty 1} \frac{dY_1}{d\xi}$

Or,

$$\frac{\partial \Phi}{\partial \eta} = \frac{A}{\beta} V_{\infty 1} \frac{dY_1}{d\xi}$$

# Boundary condition (Continued)

- At the bottom of the previous slide we get:

$$\left. \frac{\partial \Phi}{\partial \eta} \right|_{\eta=0} = \frac{A}{\beta} V_{\infty 1} \frac{dY_1}{d\xi}$$

We want to satisfy the incompressible flow boundary condition

$$\Phi_{,\eta} \Big|_{\eta=0} = V_{\infty 1} \frac{dY_1}{d\xi}$$

Comparing the two, we get  $A=\beta$

Two down, one more to go.. Expression for  $C_p$

# Recap of where we are..

- We have found all the constants that will satisfy the governing equation and boundary conditions, assuming the bodies (airfoils) in both problems (incompressible and compressible) have the same shape.
- Let us summarize what we have thus far.

$$\xi = x$$
$$\eta = \beta y$$

$$\Phi = A\varphi = \beta\varphi$$

# We finally turn to surface pressure coefficient

- All we are after is the surface pressure coefficient.
- If we know  $C_p$ , we can integrate it to get lift and moment coefficients.

$$C_p = -2 \frac{\varphi_x}{V_\infty}$$

Use the relations from the previous slide to  
Replace  $x$ ,  $\varphi$  and  $V_\infty$

$$\xi = x \quad \Phi = \beta \varphi \quad V_{\infty 1} = V_\infty$$

We get:

$$C_p = -2 \frac{\varphi_x}{V_\infty} = -\frac{2 \Phi_\xi}{\beta V_{\infty 1}} = \frac{C_{p, incompressible}}{\beta}$$

# We get Prandtl-Glauert Rule

- For the same geometry (i.e. slopes are the same for compressible and incompressible problem in the wind tunnel coordinate system at all points on the body):

$$C_p = \frac{C_{p,incompressible}}{\beta} = \frac{C_{p,incompressible}}{\sqrt{1 - M_\infty^2}}$$

# Lift and Moment Coefficients

- Integrating  $C_p$  to get  $C_l$  and  $C_m$  yields

$$C_{l,compressible} = \frac{1}{\beta} C_{l,incompressible}$$

$$C_{d,compressible} = \frac{1}{\beta} C_{d,incompressible} = 0$$

*and,*

$$C_{m,compressible} = \frac{1}{\beta} C_{m,incompressible}$$

*if the airfoil shapes are the same.*

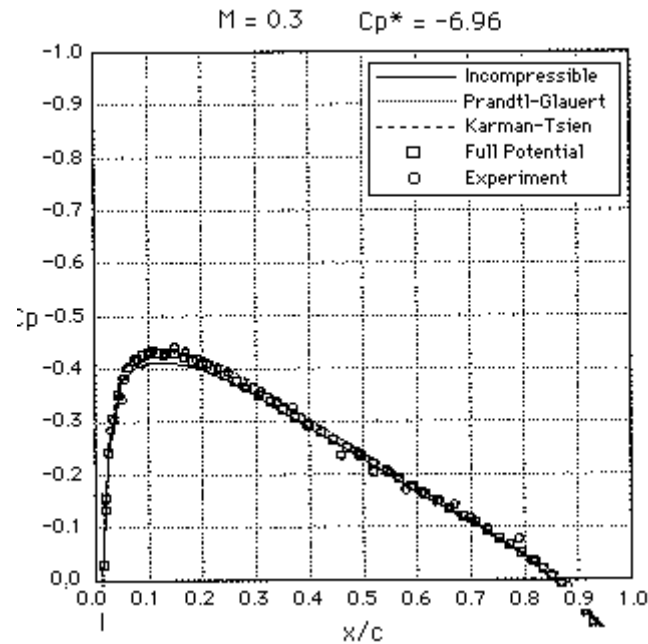
# Summary

- Analyze or test the same geometry in incompressible flow.
- Use Prandtl-Glauert relationship to link the incompressible pressure, lift, and moment coefficients to their counterparts in compressible flow.
- Our theory does not give drag. Potential flow drag is zero for 2-D, no viscous effects yet.

# Comparisons with Test data

## NACA 0012 Airfoil at zero angle of attack

<http://www.desktop.aero/appliedaero/compressibility/cpplot3.html>

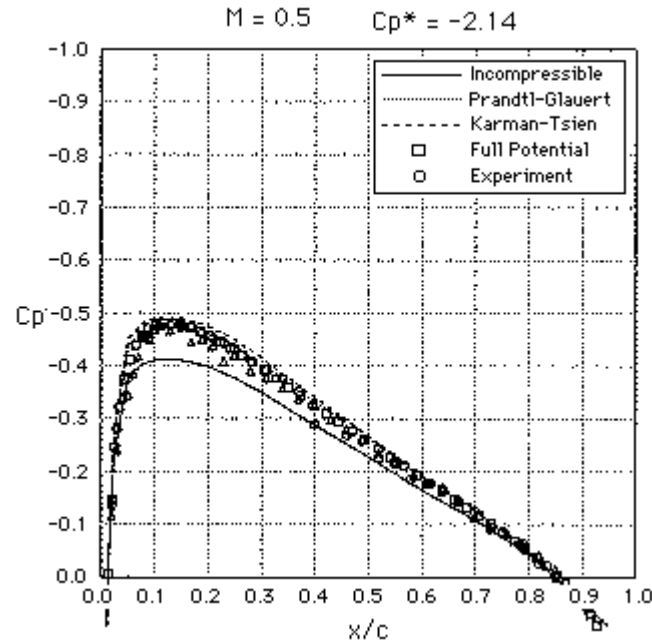


Good agreement

# Comparisons with Test data

## NACA 0012 Airfoil at zero angle of attack

<http://www.desktop.aero/appliedaero/compressibility/cpplot3.html>

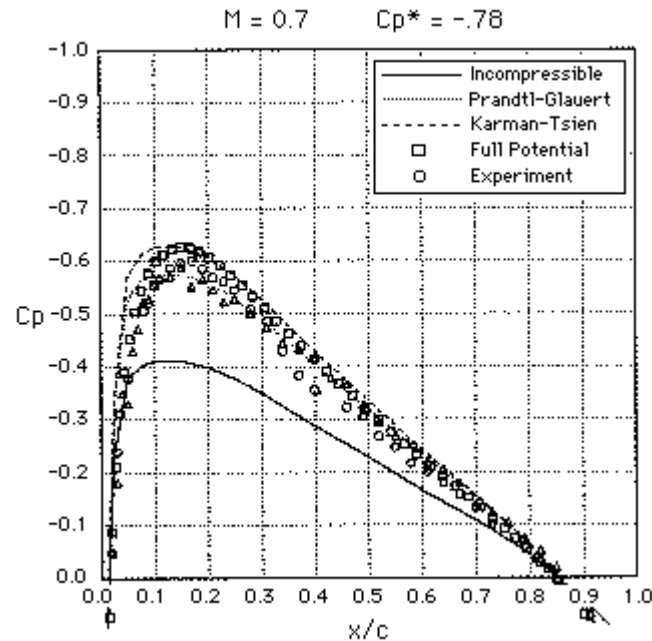


Good agreement

# Comparisons with Test data

## NACA 0012 Airfoil at zero angle of attack

<http://www.desktop.aero/appliedaero/compressibility/cpplot3.html>

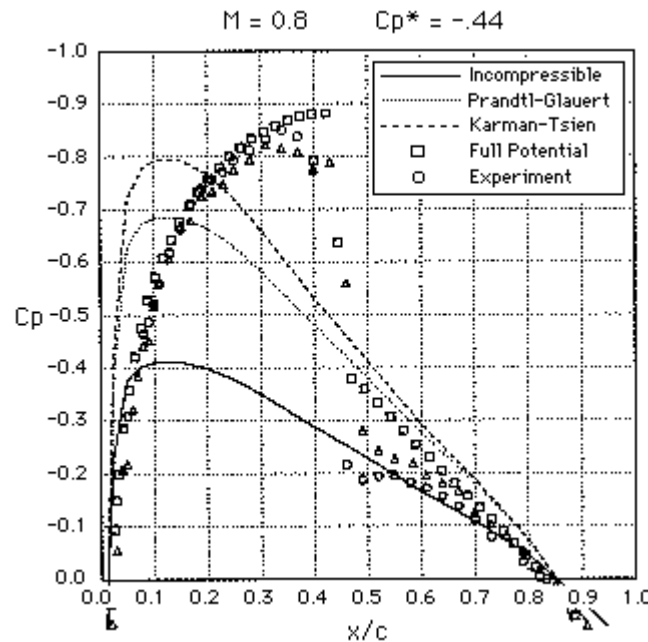


Agreement deteriorates..

# Comparisons with Test data

## NACA 0012 Airfoil at zero angle of attack

<http://www.desktop.aero/appliedaero/compressibility/cpplot3.html>



Poor agreement...

Shocks waves form

P-G Rule can not

Pick up formation of shocks.

# Other “Similarity” Rules

- Because the rules we are developing assume that the compressible flow (pressure, lift, moment) are similar to incompressible flow (pressure, lift, and drag), these rules are called “Similarity” rules in some literature.
- We have derived P-G Rule and Gothert’s Rule.
- There are two other rules, Karman-Tsien Rule and Laitone’s Rule.
  - These are more accurate at higher Mach numbers, as transonic effects become more imminent.
  - We will just state them without proving them.

# Karman-Tsien Rule

- Analyze the same body (as P-G Rule does) in incompressible flow. Compute  $C_p$ . We call this incompressible  $C_p$  (at Mach=0) the symbol  $C_{p,0}$  in the equation below.

$$C_p = \frac{C_{p,0}}{\sqrt{1 - M_\infty^2}} + \frac{M_\infty^2}{1 + \sqrt{1 - M_\infty^2}} \frac{C_{p,0}}{2}$$

Incompressible  $C_p$  for same body, same alpha

Compressible flow  $C_p$

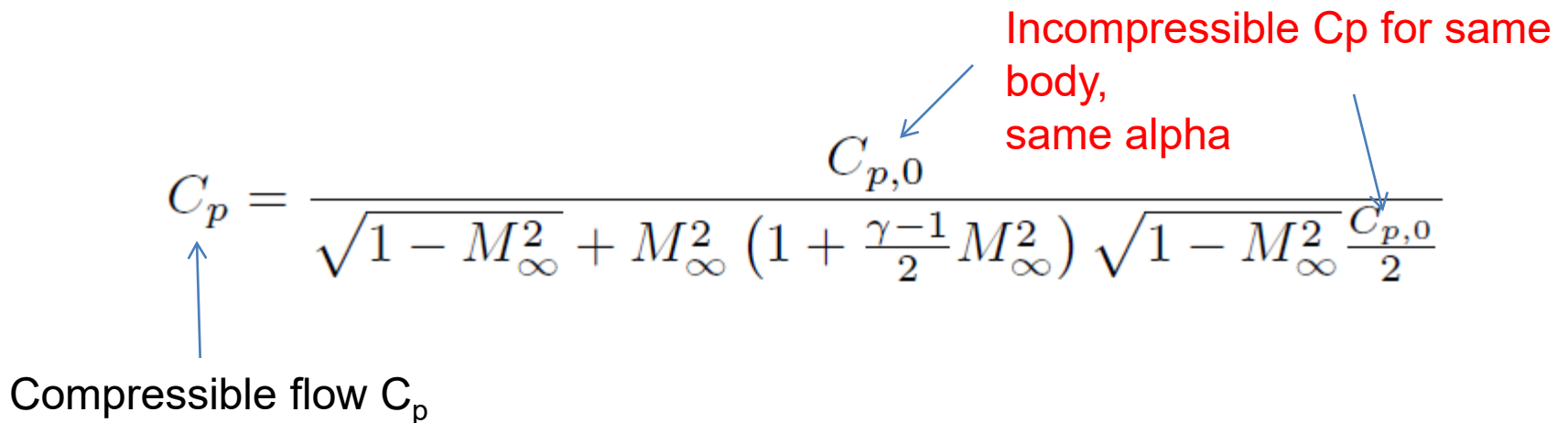
# Laitone's Rule

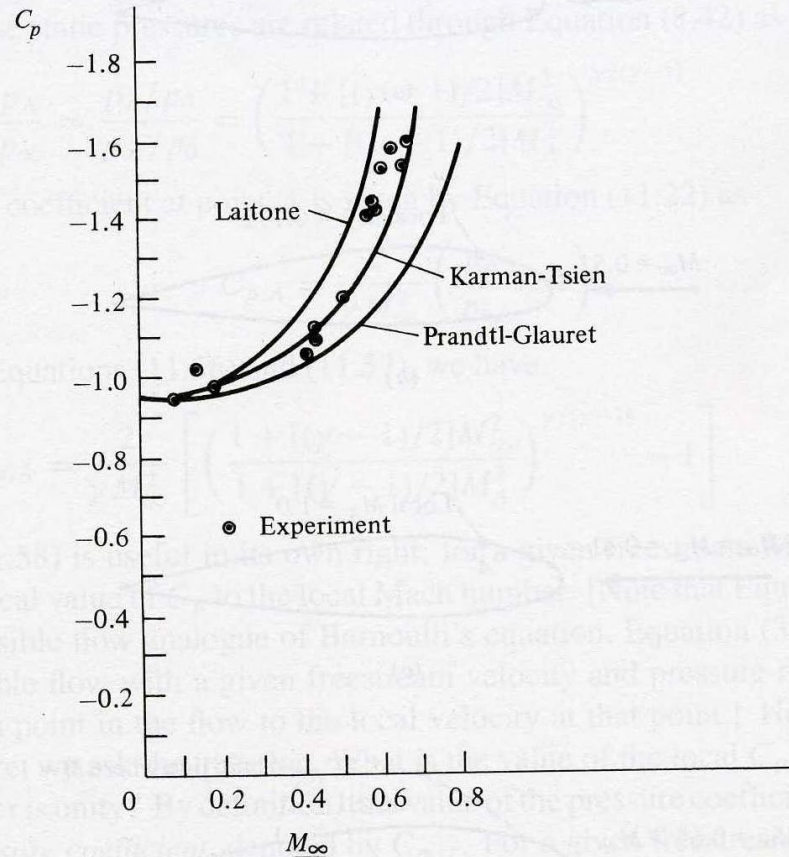
- Analyze or test same body, at same alpha in the incompressible flow.

$$C_p = \frac{C_{p,0}}{\sqrt{1 - M_\infty^2} + M_\infty^2 \left(1 + \frac{\gamma - 1}{2} M_\infty^2\right) \sqrt{1 - M_\infty^2} \frac{C_{p,0}}{2}}$$

Compressible flow  $C_p$

Incompressible  $C_p$  for same body, same alpha

The diagram shows the equation for Laitone's Rule. A blue arrow points from the text 'Compressible flow Cp' to the Cp on the left side of the equation. Another blue arrow points from the text 'Incompressible Cp for same body, same alpha' to the Cp,0 terms in the numerator and denominator of the fraction. The text 'Incompressible Cp for same body, same alpha' is written in red.



**Figure 11.4**

Several compressibility corrections compared with experimental results for an NACA 4412 airfoil at an angle of attack  $\alpha = 1^\circ 53'$ . The experimental data are chosen for their historical significance; they are from NACA report no. 646, published in 1938 (Reference 30). This was the first major NACA publication to address the compressibility problem in a systematic fashion; it covered work performed in the 2-ft high-speed tunnel at the Langley Aeronautical Laboratory and was carried out during 1935–1936.

**Example 11.1**

At a given point on the surface of an airfoil, the pressure coefficient is  $-0.3$  at very low speeds. If the freestream Mach number is  $0.6$ , calculate  $C_p$  at this point.

**Solution**

From Equation (11.51),

$$C_p = \frac{C_{p,0}}{\sqrt{1 - M^2}} = \frac{-0.3}{\sqrt{1 - (0.6)^2}} = \boxed{-0.375}$$

**Example 11.2**

From Chapter 4, the theoretical lift coefficient for a thin, symmetric airfoil in an incompressible flow is  $c_l = 2\pi\alpha$ . Calculate the lift coefficient for  $M_\infty = 0.7$ .

**Solution**

From Equation (11.52),

$$c_l = \frac{c_{l,0}}{\sqrt{1 - M_\infty^2}} = \frac{2\pi\alpha}{\sqrt{1 - (0.7)^2}} = \boxed{8.8\alpha}$$

*Note:* The effect of compressibility at Mach  $0.7$  is to increase the lift slope by the ratio  $8.8/2\pi = 1.4$ , or by 40 percent.

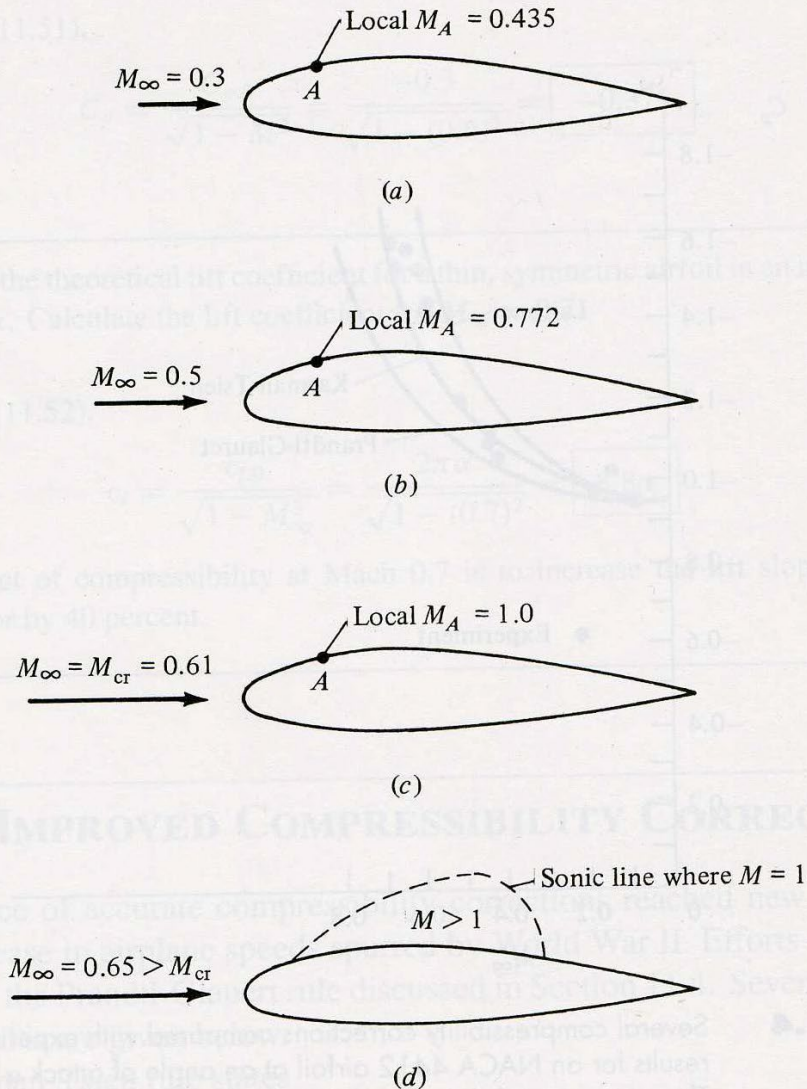
# Critical Mach Number

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# Preliminary Remarks

- As the freestream Mach number increases from low subsonic flow to higher values, at some point the flow will first become supersonic somewhere on the airfoil (just outside the boundary layer).
- This Mach number is called “Critical Mach number.”
- Above this Mach number, pockets of supersonic flow, terminated by shock waves will occur.
- Shocks are undesirable – they cause rise in drag, and may cause the boundary layer to separate and airfoil to stall.
- For this reason, it is important to determine (or accurately estimate) the Critical Mach number .
  - Critical mach number is a function of the geometry and the angle of attack (or  $C_l$ ) at which the airfoil operates.
  - If the body is thicker, has more camber, or has a higher alpha, then the flow will speed up (or be disturbed) more, and critical mach number will be reached sooner.
- Some propeller and helicopter blades become thinner near the tip to avoid the onset of transonic flow, and increase the critical mach number to as high a value as possible.



Critical Mach number  
for this airfoil  
at this alpha is  
0.61

**Figure 11.5** Definition of critical Mach number. Point A is the location of minimum pressure on the top surface of the airfoil. (See end-of-chapter Problem 11.7 for the calculation of the numbers in this figure.)

# We start with Pressure Coefficient at a point A on the airfoil

$$C_p = \frac{\frac{p_A}{p_\infty} - 1}{\frac{\gamma}{2} M_\infty^2}$$

See earlier lecture #4

We use isentropic law linking p and T:  $\frac{p_A}{p_\infty} = \left(\frac{T_A}{T_\infty}\right)^{\frac{\gamma}{\gamma-1}}$

We next use energy equation to replace T with M:

$$\frac{T_0}{T_A} = 1 + \frac{\gamma-1}{2} M_A^2$$
$$\frac{T_0}{T_\infty} = 1 + \frac{\gamma-1}{2} M_\infty^2$$

# Pressure Coefficient, Continued..

- The energy equation yields (divide one by the other):

$$\frac{T_0}{T_A} = 1 + \frac{\gamma-1}{2} M_A^2 \quad \text{becomes} \quad \frac{T_A}{T_\infty} = \frac{1 + \frac{\gamma-1}{2} M_\infty^2}{1 + \frac{\gamma-1}{2} M_A^2}$$
$$\frac{T_0}{T_\infty} = 1 + \frac{\gamma-1}{2} M_\infty^2$$

Use this in isentropic gas law from previous slide, and plug the result into expression for  $C_p$ . After very minor algebra, we get any point A on the airfoil:

$$C_{p,A} = \frac{2}{\gamma M_\infty^2} \left( \left( \frac{1 + \frac{\gamma-1}{2} M_\infty^2}{1 + \frac{\gamma-1}{2} M_A^2} \right)^{\frac{\gamma}{\gamma-1}} - 1 \right)$$

# Critical mach Number (Continued)

- The equation at the bottom of the previous slide determines how  $C_p$  varies with local mach number.

$$C_{p,A} = \frac{2}{\gamma M_\infty^2} \left( \left( \frac{1 + \frac{\gamma-1}{2} M_\infty^2}{1 + \frac{\gamma-1}{2} M_A^2} \right)^{\frac{\gamma}{\gamma-1}} - 1 \right)$$

When critical Mach number is achieved, at some point on the airfoil  $M$  will become 1. Then, at this point on the body,

$$C_{p,cr} = \frac{2}{\gamma M_{cr}^2} \left( \left( \frac{1 + \frac{\gamma-1}{2} M_{cr}^2}{1 + \frac{\gamma-1}{2}} \right)^{\frac{\gamma}{\gamma-1}} - 1 \right)$$

# Cp at Critical Mach Number

- From P-G Rule, Cp is related to incompressible  $C_{p,0}$  by the factor  $\beta$ .
- At critical mach number , P-G Rule gives:

$$C_{p,cr} = \frac{C_{p,incompressible}}{\beta_{cr}} = \frac{C_{p,incompressible}}{\sqrt{1 - M_{cr}^2}}$$

Equate this to expression we got from the previous slide:

$$C_{p,cr} = \frac{2}{\gamma M_{cr}^2} \left( \left( \frac{1 + \frac{\gamma-1}{2} M_{cr}^2}{1 + \frac{\gamma-1}{2}} \right)^{\frac{\gamma}{\gamma-1}} - 1 \right)$$

At critical Mach Number, at the point on the airfoil where local  $M=1$  we get

$$\frac{C_{p,incompressible}}{\sqrt{1-M_{cr}^2}} = \frac{2}{\gamma M_{cr}^2} \left[ \left( \frac{1 + \frac{\gamma-1}{2} M_{cr}^2}{1 + \frac{\gamma-1}{2}} \right)^{\frac{\gamma}{\gamma-1}} - 1 \right]$$

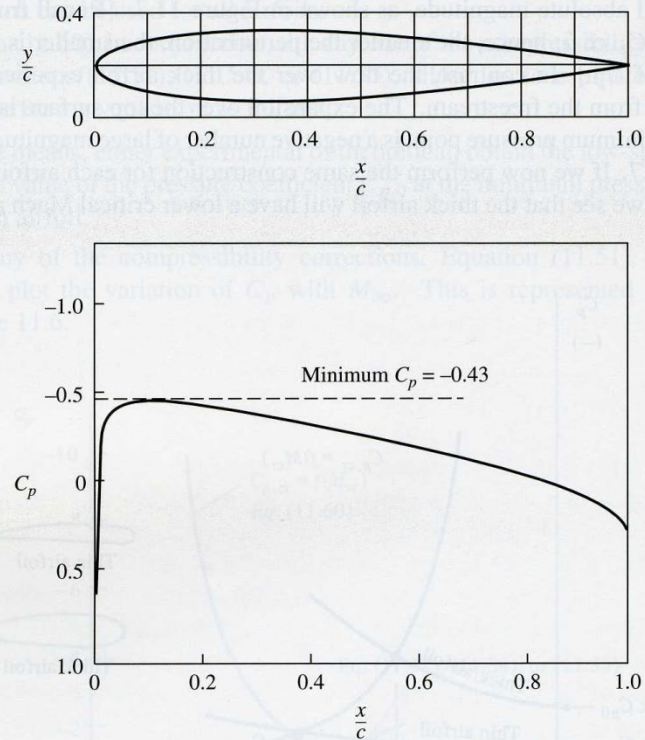
We will solve this equation, usually numerically, to get the critical mach number  
Given the lowest negative  $C_p$  under incompressible flow conditions on the airfoil.

Why lowest negative  $C_p$ ? This is where, by Bernoulli's equation, we will get the  
Highest velocity in incompressible flow.

The point on the airfoil where the highest incompressible flow velocity  
Occurs is also the point where the sonic conditions are first likely to  
Occur.

**Example 11.3**

In this example, we illustrate the estimation of the critical Mach number for an airfoil using (a) the graphical solution discussed in this section, and (b) an analytical solution using a closed-form equation obtained from a combination of Equations (11.51) and (11.60). Consider the NACA 0012 airfoil at zero angle of attack shown at the top of Figure 11.8. The pressure coefficient distribution over this airfoil, measured in a wind tunnel at low speed, is given at the

**Figure 11.8**

Low-speed pressure coefficient distribution over the surface of an NACA 0012 airfoil at zero angle of attack.  $Re = 3.65 \times 10^6$ . (Source: R. J. Freuler and G. M. Gregorek, "An Evaluation of Four Single Element Airfoil Analytical Methods," in *Advanced Technology Airfoil Research*, NASA CP 2045, 1978, pp. 133–162.)

## Solution

(a) *Graphical Solution.* First, let us accurately plot the curve of  $C_{p,cr}$  versus  $M_{cr}$  from Equation (11.60),

$$C_{p,cr} = \frac{2}{\gamma M_{cr}^2} \left[ \left( \frac{1 + [(\gamma - 1)/2] M_{cr}^2}{1 + (\gamma - 1)/2} \right)^{\gamma/(\gamma-1)} - 1 \right] \quad [11.60]$$

For  $\gamma = 1.4$ , from Equation (11.60) we can tabulate

$M_\infty$	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$C_{p,cr}$	-3.66	-2.13	-1.29	-0.779	-0.435	-0.188	0

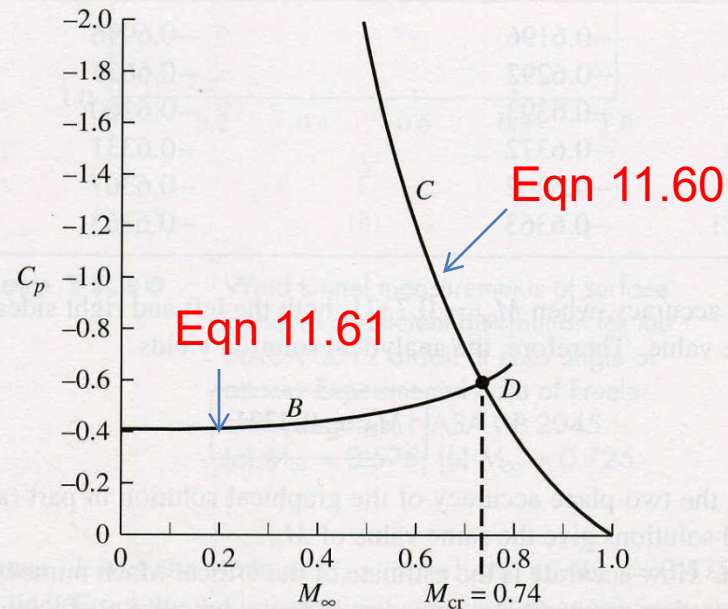
These numbers are plotted as curve  $C$  in Figure 11.9. Note that  $C_{p,cr} = 0$  when  $M_{cr} = 1.0$ . This makes physical sense; if the free stream Mach number is already 1, then no change in the pressure is required to achieve Mach 1 at a local point in the flow, and hence the pressure difference ( $p_{cr} - p_\infty$ ) is zero and  $C_{p,cr} = 0$ .

$$(C_p)_{\min} = \frac{(C_{p,0})_{\min}}{\sqrt{1 - M_\infty^2}} = \frac{-0.43}{\sqrt{1 - M_\infty^2}}$$

**[11.61]**

Some values of  $(C_p)_{\min}$  are tabulated below

$M_\infty$	0	0.2	0.4	0.6	0.8
$(C_p)_{\min}$	-0.43	-0.439	-0.469	-0.538	-0.717



**Figure 11.9** Graphical solution for the critical Mach number.

Where these two curves cross gives the critical mach number of 0.74.

We can also solve for Critical Mach number numerically..  
By trial and mostly, error

$$\frac{-0.43}{\sqrt{1 - M_{cr}^2}} = \frac{2}{\gamma M_{cr}^2} \left[ \left( \frac{1 + \frac{\gamma - 1}{2} M_{cr}^2}{1 + \frac{\gamma - 1}{2}} \right)^{\frac{\gamma}{\gamma - 1}} - 1 \right]$$

Plug in estimates of  $M_{cr}$  on left and right, see when the equality holds.

As our estimate for  $M_{cr}$  increases, the left side will increase in magnitude, right side will decrease in magnitude.

# Trial and error approach

$$M_{cr} \quad \frac{-0.43}{\sqrt{1 - M_{cr}^2}} \quad \frac{2}{\gamma M_{cr}^2} \left[ \left( \frac{1 + [(\gamma - 1)/2] M_{cr}^2}{1 + (\gamma - 1)/2} \right)^{\gamma/\gamma - 1} - 1 \right]$$

---

0.72	-0.6196	-0.6996
0.73	-0.6292	-0.6621
0.74	-0.6393	-0.6260
0.738	-0.6372	-0.6331
0.737	-0.6362	-0.6367
0.7371	-0.6363	-0.6363

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**Figure 11.10**

Wind tunnel measurements of surface pressure coefficient distribution for the NACA 0012 airfoil at zero angle of attack. Experimental data of Frueler and Gregorek, NASA CP 2045.  
(a)  $M_\infty = 0.575$ , (b)  $M_\infty = 0.725$ .

