Chapter III

Part I: Rotational vs. Irrotational Flow, Circulation, Stream Function, Potential Flow

From now on, we will only deal with steady, incompressible, inviscid flow. We will neglect all body forces, even gravity.

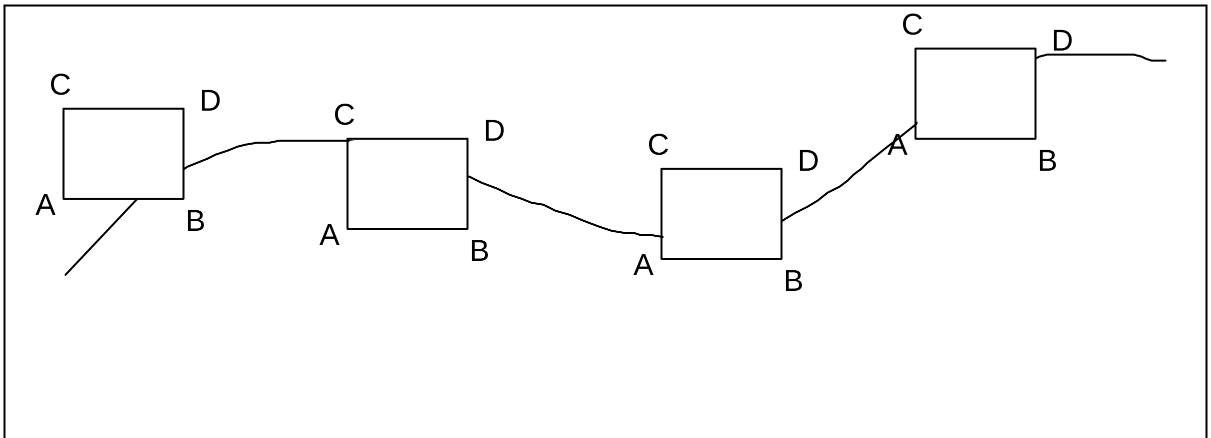
In the previous chapter, we derived an expression for vorticity vector:

$$\boxed{\vec{x} = \vec{\nabla} \times \vec{V}} \tag{1}$$

We mentioned that vorticity is a measure of the angular velocity of the fluid elements.

A flow in which there is a lot of vorticity is called a <u>rotational flow</u>. In rotational flows, the fluid elements will rotate as they move from upstream to downstream. This is like a snow ball tumbling down a steep hill side.

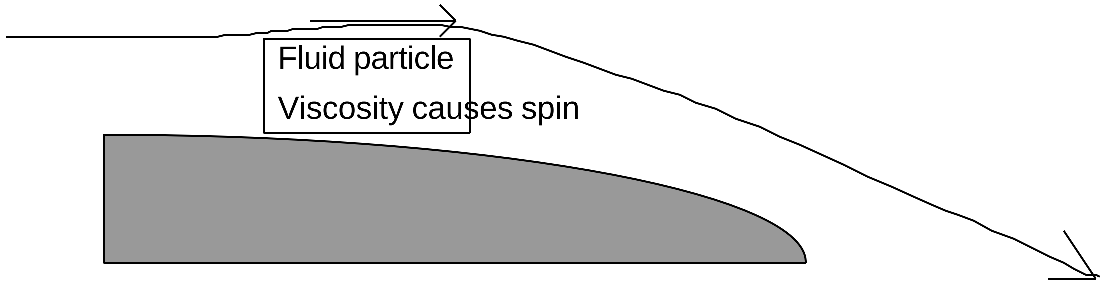
A flow in which there is very little vorticity is called an <u>irrotational</u> flow. The fluid particles may move up and down, laterally etc. However, like boxes placed on an escalator, they will maintain their upright position (whatever it is) as they move from point to point.



In irrotational flow, the fluid elements do not spin about, but maintain their upright orientation.

What causes fluid particles to rotate? A number of physical phenomena can. Here are a few:

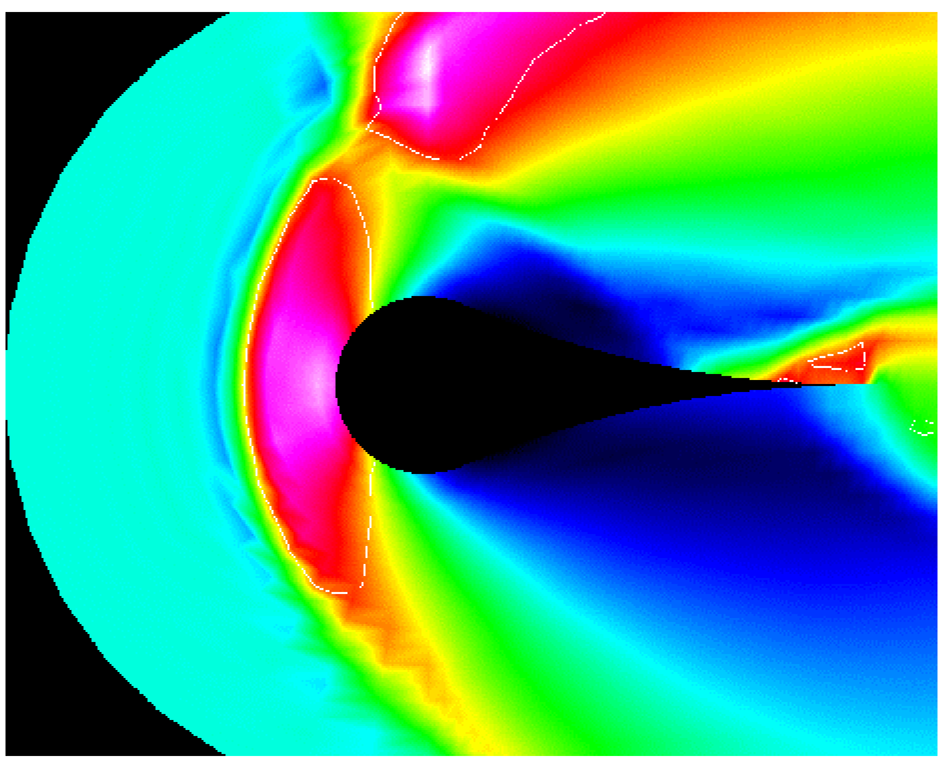
1. Viscous stresses can pull the fluid particles in different direction. Consider a fluid particle right next to a solid surface. The bottom side of the particle may want to stick to the surface due to viscosity. The top side of the particle may be dragged by the particles above in the general direction of the flow. The result is a rotation of the particle.



Thus, boundary layers and viscous regions are rotational, and filled with vorticity.

2. Velocity differences at the edge of a jet can cause the top part of a fluid particle to move faster than the lower part, or vice versa. Result: Rotation.

3. Shock waves can slow down the flow. The strength of shock waves depends on the angle they make with the flow. There is usually a curved "bow shock" ahead of a blunt body. Even though the fluid particles ahead of the shock may be uniform, different fluid particles may impact the shock at different angles, and slowed down differently behind the shock wave. Result: rotation.



4. Non-uniform body forces (example, forces exerted by a spinning propeller on air particles) can cause different fluid particles to be accelerated differently, causing a non-uniform velocity field. Result: Rotation.

Although most flows encountered in nature are rotational, if we stay away from viscous regions (boundary .layers), curved shock waves or non-uniform body forces, in all other regions the flow may be assumed to be irrotational. Assuming the flow to be irrotational greatly simplifies the mathematics, and the eventual solution of the flow on the computer. We will be invoking the irrotational flow assumption soon.

Circulation: Consider a closed curve C in space. It is made of tiny straight line segments 'ds' each which may be thought of as a short vector. Then circulation over this contour C is defined as:

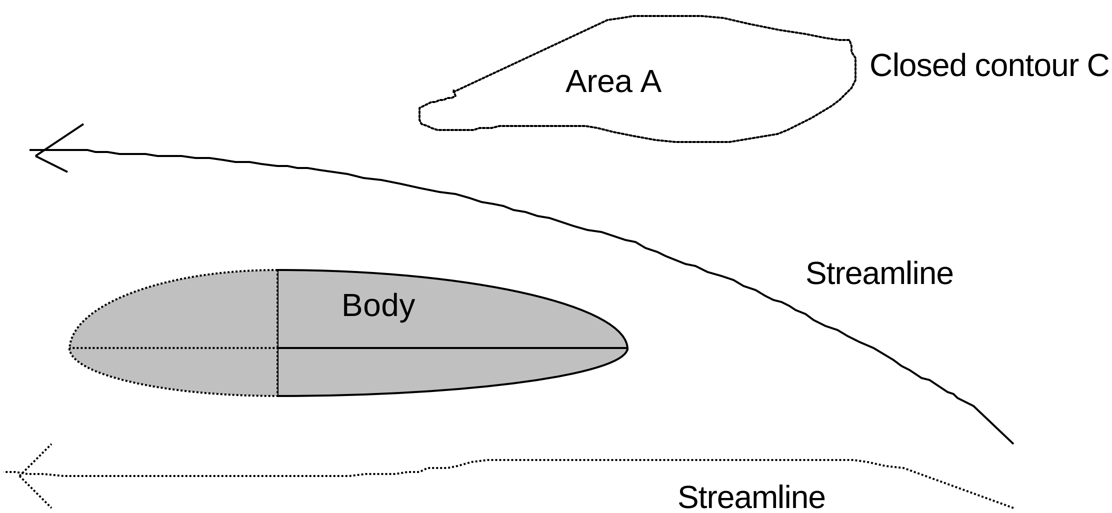
$$\boxed{\Gamma = \oint_C \vec{V} \bullet d\vec{s}} \tag{2}$$

There is a theorem called Stokes Theorem which states that the circulation is related to the vorticity vectors crossing the surface A enclosed by the closed curve C as follows:

$$\Gamma = \oint_C \vec{V} \bullet d\vec{s} = \iint_S \vec{x} \bullet d\vec{A} \tag{3}$$

The area A may be assumed to be made of several infinitesimal elements dA. The vector $d\vec{A}$ has the magnitude of the area dA, and is directed normal to dA.

Stokes theorem says that if there is circulation, then there is vorticity, and vice versa. There will be no circulation over a contour C if the area captured by that contour does not contain any vorticity.



What happens if the contour C encloses the body? Then, since the body has a boundary layer, and the boundary layer contains vorticity, the circulation over any contour that encloses the body will usually be nonzero. The exception is a symmetric body (e.g. symmetric airfoil) which contains clockwise vorticity on one side, say the upper side boundary layer and counterclockwise vorticity on the lower side.

<u>Potential Flow</u>: A irrotational flow (that is, a flow in which vorticity is zero) is also called a potential flow. This is because we can define a function called the velocity potential Φ such that

$$\boxed{\begin{aligned} &\vec{V} = \vec{\nabla} f \\ &Or, \\ &u = \frac{\partial f}{\partial x} \\ &v = \frac{\partial f}{\partial y} \\ &w = \frac{\partial f}{\partial z} \end{aligned}} \tag{4}$$

If u, v and w are defined as derivatives of Φ as shown, then vorticity in that flow is zero. This is because

$$\vec{x} = \vec{\nabla} \times \vec{V} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix} = \vec{i}\left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}\right) + \vec{j}\left(\frac{\partial^2 f}{\partial z \partial x} - \frac{\partial^2 f}{\partial x \partial z}\right) + \vec{k}\left(\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x}\right) = 0 \tag{5}$$

If we use equation (4) in continuity equation

$$\begin{aligned} &\vec{\nabla} \bullet \vec{V} = 0 \\ &Or, \\ &\left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial x} + \vec{k}\frac{\partial}{\partial z}\right) \bullet \left(\vec{i}\frac{\partial f}{\partial x} + \vec{j}\frac{\partial f}{\partial x} + \vec{k}\frac{\partial f}{\partial z}\right) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0 \end{aligned} \tag{6}$$

Or,

$$\boxed{\nabla^2 f = 0} \tag{7}$$

The operator ∇^2 is called the <u>Laplacian operator</u>. It is simply three second derivatives added together as shown below:

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \tag{8}$$

Equation (7) is called Laplace's equation for the velocity potential Φ.

Why did we introduce a new variable called Φ? It is because we would rather solve a single linear PDE for Φ (given by equation 7) than solve 4 nonlinear PDEs - conservation of mass, u- v-, w-momentum etc. If we can somehow solve equation (7), we can find u, v and w from equation (4). Finally, we can find p from the Bernoulli equation.

We will spend much of our remaining time in this course solving equations (7), (4) and the Bernoulli equation. This simple equation may be used to model flow over airfoils, wings, bodies etc. It is truely useful. Of course, it works only for inviscid, incompressible, irrotational flows. It can not be used in regions of large vorticity, e.g. boundary layers.

<u>Stream Function</u>: A less useful new variable is called stream function ψ. **It exists only in 2-D flows**, or axi-symmetric flows (e.g. flow around a body of revolution such as a sphere or a bullet). The stream function is defined as:

$$\boxed{\begin{aligned} u &= \frac{\partial y}{\partial y} \\ v &= -\frac{\partial y}{\partial x} \end{aligned}} \tag{9}$$

We define u and v this way to automatically satisfy the continuity equation for 2-D incompressible flows.

$$\boxed{\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0} \tag{10}$$

For 2-D flows we only need to worry about the z- component of vorticity given by $\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$. You may recall that we derived this component by considering the angular velocity of a fluid element in a previous hand-out titled "Pathlines, Streamlines, Vorticity and Bernoulli's Equation". In irrotational flow, vorticity is zero. Setting $\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$ to zero, and using the definitions given in equation (9) we get:

$$\boxed{\nabla^2 y = 0} \tag{11}$$

Equation (11) is Laplace's equation for stream function. It is linear, and easy enough to solve. After we somehow solve for ψ we can get u and v from equation (9). We can finally compute p from Bernoulli's equation.

Equation of a Streamline:

Recall that a streamline is defined as:

$$\frac{dx}{u} = \frac{dy}{v}$$
$$Or,$$
$$udy - vdx = 0 \qquad (12)$$

Use the definitions of u and v from equation (9). Then equation (12) becomes:

$$\boxed{\begin{array}{l}\frac{\partial y}{\partial y}dy + \frac{\partial y}{\partial x}dx = 0 \\ Or, \\ dy \;= 0\end{array}} \qquad (13)$$

Equation 913) implies that ψ is constant along a streamline.

We have thus two ways of solving 2-D inviscid irrotational flows (potential flows, for short)- solve for ψ or solve for Φ. Both require solving the Laplace's equation.

We have only a single way of solving 3-D inviscid, irrotational flow (potential flows, for short). Solve for Φ. The stream function ψ is not defined in 3-D flows.

AE 3003 Chapter III - Part 2
Sources, Sinks and Doublets - the Building Blocks of Potential Flow

In the previous handout we developed the following equation for the velocity potential:

$$\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2 f}{\partial z^2}=0$$
$$Or$$
$$\nabla^2 f=0 \tag{14}$$

where the operator ∇^2 is called the Laplacian operator. This equation holds for 2-D and 3-D inviscid irrotational flows. If we are only interested in 2-D irrotational inviscid flows, we may also solve for:

$$\nabla^2 y=0 \tag{15}$$

where y is the stream function.

After we have solved for the velocity potential or the stream function, we can compute the velocities. In a Cartesian coordinate system, for 2-D flows, we will use:

$$u=\frac{\partial f}{\partial x}=\frac{\partial y}{\partial y}$$
$$v=\frac{\partial f}{\partial y}=-\frac{\partial y}{\partial x} \tag{16}$$

In a polar coordinate system, for 2-D flows we will use:

$$v_r=\text{Radial velocity}=\frac{\partial f}{\partial r}=\frac{1}{r}\frac{\partial y}{\partial q}$$
$$v_q=\text{Tangential velocity}=\frac{1}{\mathrm{r}}\frac{\partial f}{\partial q}=-\frac{\partial y}{\partial r} \tag{17}$$

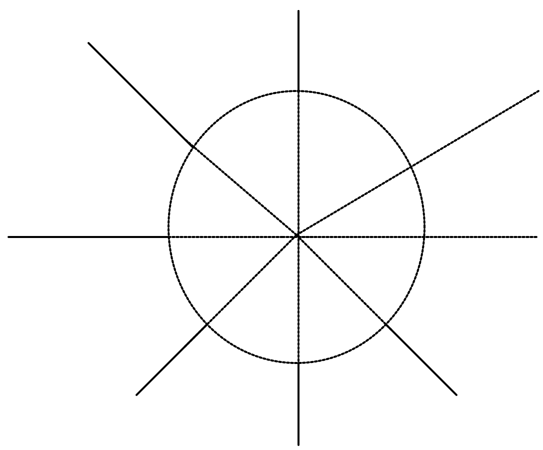
In 3-D, the velocities are given only in terms of the velocity potential, as follows:

$$\vec{V}=\vec{\nabla} f$$
$$Or,$$
$$u=\frac{\partial f}{\partial x} \qquad v=\frac{\partial f}{\partial y} \qquad w=\frac{\partial f}{\partial z} \tag{18}$$

Once the velocity is known, we can find pressure from the Bernoulli's equation.

In this section, we consider some simple solutions to the Laplace's equation (14) or (15). Since these equations are linear, we can superpose many such simple solutions to arrive at a more complex flow field. This is like building a complex configuration using Lego blocks. The individual simple solutions are the individual Lego pieces, which on their own, are not very interesting. Together, however, they can solve some very interesting flows, including flow over airfoils and wings.

<u>Building Block #1: 2-D Sources and Sinks</u>: A source is like a lawn sprinkler. It sprays the water (or air) radially, and equally, in all the directions, at the rate of Q units per unit time. If this is a sink (e.g. a drain hole on a concrete pavement) the velocity vectors will still be radial, but directed inwards towards the center. The sign of Q will be positive for a source, and negative for a sink.



Consider a circle of radius r enclosing this source. Let v_r be the radial component of velocity associated with this source (or sink). Then, form conservation of mass, for a cylinder of radius r, and unit height perpendicular to the paper,

$$Q = (2pr)\cdot(1)\cdot v_r$$
$$Or,$$
$$v_r = \frac{Q}{2pr} \qquad (19)$$

Solving equation (19) for the velocity potential and the stream function we get, for a source or a sink:

$$\boxed{\begin{aligned} f &= \frac{Q}{2p}\log_e r \\ y &= \frac{Q}{2p} q \end{aligned}} \qquad (20)$$

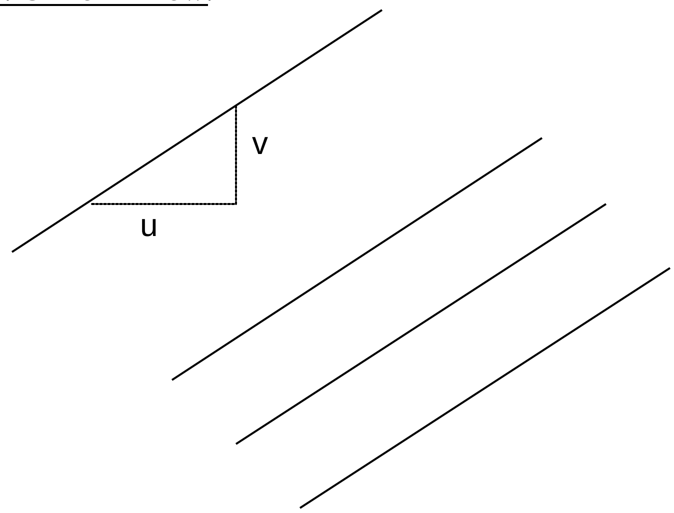
plus a constant. The constants appear as we integrate the velocity to get the velocity potential or stream function. Every simple solution we consider will be an analytical function plus a constant to be determined later.

Exercise: Verify for yourselves that (20) satisfies Laplace's equation in polar coordinates:

$$\nabla^2 f = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial q^2} = 0$$

$$\nabla^2 f = \frac{\partial^2 y}{\partial r^2} + \frac{1}{r}\frac{\partial y}{\partial r} + \frac{1}{r^2}\frac{\partial^2 y}{\partial q^2} = 0$$

Building Block #2: Uniform Flow:



This is also, by itself, an uninteresting flow. It represents a uniform flow with the velocity components u_∞ and v_∞ along the x- and y- axes. The stream function and the velocity potential associated with this flow are:

$$f_{\text{Uniform Flow}} = u_\infty x + v_\infty y$$
$$y_{\text{Uniform Flow}} = u_\infty y - v_\infty x \qquad (21)$$

If we use equation (21) on the definitions given in equation (16), we recover the Cartesian components of velocity. Notice that these functions are simple straight lines. It is also easy to see that these functions given in equation (21) satisfy the Laplace's equation.

Superposition of a Source and a Uniform Flow:

Let us try to superpose the uniform flow and the flow field due to a source. All we have to do is add the flow field given in equations (21) to equation (20). The result is given below:

$$f = \frac{Q}{2p}\log_e r + u_\infty x + v_\infty y$$
$$y_{\text{Uniform Flow}} = u_\infty y - v_\infty x + \frac{Q}{2p} q \qquad (22)$$

We can take x and y- derivatives of this flow field to get the velocity field. Note that the quantity 'r' represents the distance between where the source (x_s,y_s), and a general point (x,y) where the distance is being computed. That is,

$$r = \sqrt{(x - x_{Source})^2 + (y - y_{Source})^2} \tag{23}$$

Thus, the velocity potential in Cartesian form, taking into account where the source has been placed, is:

$$f = \frac{Q}{4p}\log\left[(x - x_s)^2 + (y - y_s)^2\right] + u_\infty x + v_\infty y \tag{24}$$

and the velocities are:

$$u = \frac{\partial f}{\partial x} = \frac{Q}{8p}\frac{(x - x_s)}{(x - x_s)^2 + (y - y_s)^2} + u_\infty$$
$$v = \frac{\partial f}{\partial y} = \frac{Q}{8p}\frac{(y - y_s)}{(x - x_s)^2 + (y - y_s)^2} + v_\infty \tag{25}$$

These velocities may be plugged into the Bernoulli's equation:

$$p + \frac{1}{2}\left(u^2 + v^2\right) = p_\infty + \frac{1}{2}\left(u_\infty^2 + v_\infty^2\right) \tag{26}$$

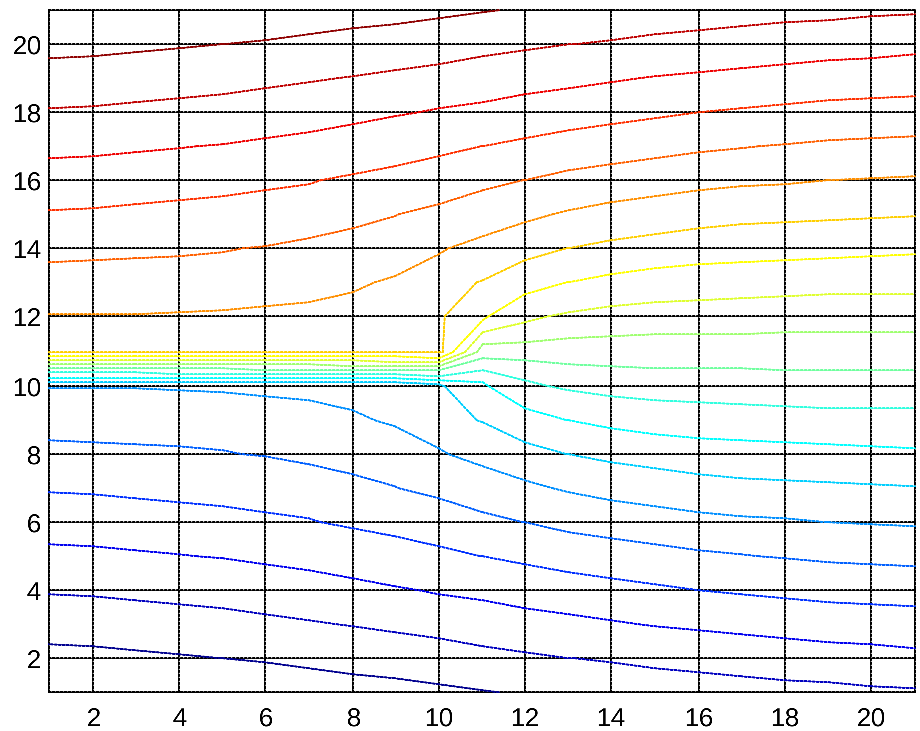
where p_∞ is the pressure value far away from the source.

We can also plot the flow field and the streamlines. The easiest way to accomplish this is using built-in functions such as the MATLAB function "contour". Here is an example. In this example, a source of strength Q equal to unity is placed at the origin. The freestream velocity is UINF= 1, and VINF=0. Here is the MATLAB script for modeling this flow.

```
X=-1:.1:1;
Y=X;
[x,y] = meshgrid(X,Y);

 Q=1;
 UINF=1.;
 VINF=0.;
 z=Q/(2.*3.14158)*atan2(y,x)+UINF*y-VINF*x;
 contour(z,20);
```

This example produces the contours shown below. Note that this looks like flow around the nose of a body. This body is called "Rankine's Half-Body."



<u>Superposition of Uniform flow, source and a sink</u>:

We can superpose a source placed at (X1,Y1), a sink of equal strength placed at (X2,Y2), and a uniform velocity. Let us say, for the sake of illustration, that the source is placed at x=-0.3, and the sink is placed at x= +0.3. on the x- axis. Let us assume that Q=1, UINF=1 and VINF=0.0. Then we can look at the contours by executing the following MATLAB script:

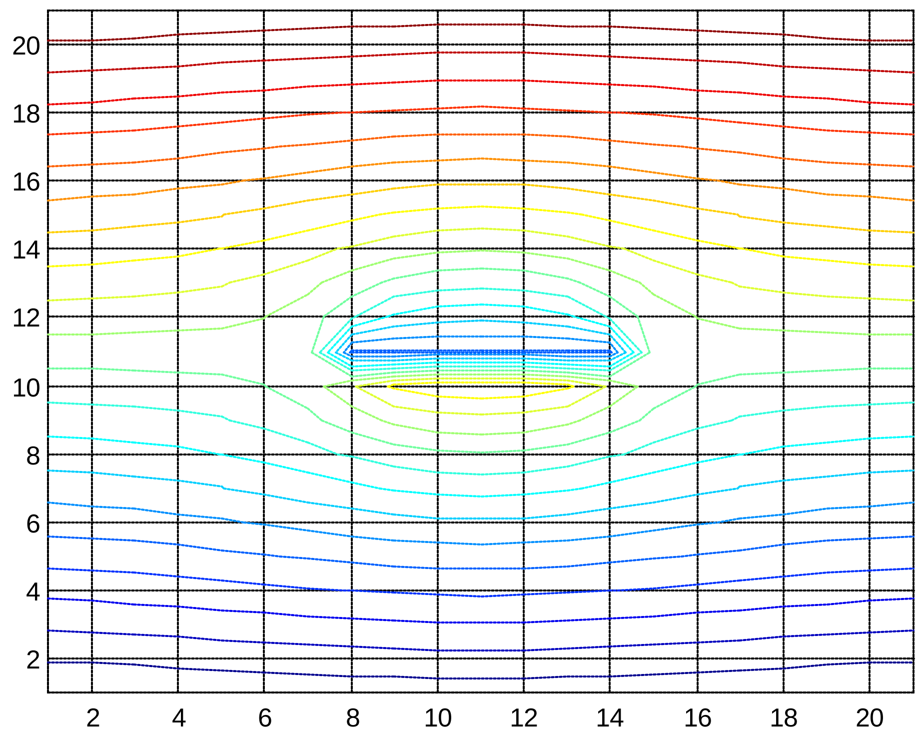
```
%Contours of  Stream function caused by a source of
strength Q placed in a
%         uniform stream with Cartesian components u=UINF
and v=VINF.
% The source is placed in this example at X= -0.3, and the
sink is
% placed at x= +.3

X=-1:.1:1;
Y=X;
        [x,y] = meshgrid(X,Y);

 Q=1;
 UINF=1.;
 VINF=0.;
```

```
 z=Q/(2.*3.14158)*atan2(y,x+.3)-Q/(2.*3.14158)*atan2(y,x-
.3)+UINF*y-VINF*x;
 contour(z,20);
```

Here are the contours from the resulting plot. This flow resembles flow over an oval shaped object, called "Rankine's full body". It is similar to the shape made popular in Ford commercials.



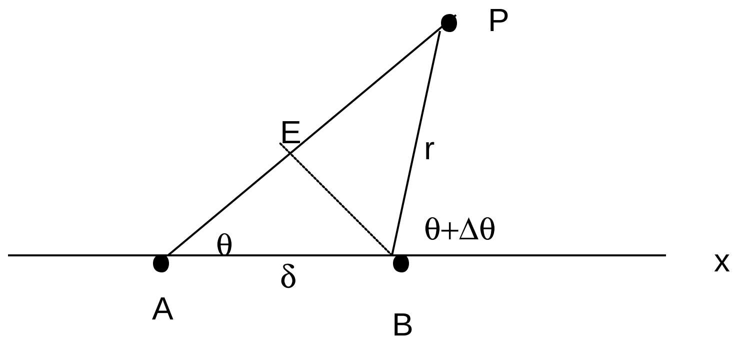
<u>Doublets</u>:

Doublets are source-sink pairs, initially separated by a distance δ, which are brought close together by making the separation distance δ →0. To keep them from annihilating each other, their strength Q is progressively increased so that Q times δ remains a constant. This constant is given the symbol μ, called the strength of the doublet.

We can derive expressions for the stream function (and the velocity potential) for a doublet from the known expressions for sources and sinks. Consider a source of strength Q placed on the x-axis at a point A, and a sink of strength –Q placed on the x-axis at point B. The points A and B are placed a distance δ apart. Then, the stream function at a general point P in the flow field is given by:

$$\psi_{Doublet} = \underset{\delta \to 0}{Limit} \frac{Q}{2\pi}\left(\theta_1 - \theta_2\right)$$

where θ_1 is the angle formed by the line AP with respect to the x- axis, and θ_2 is the angle formed by the line BP with respect to the x- axis. See the figure below.



In the figure above, the angle EPB is $(\theta_2 - \theta_1) = \Delta\theta$.

The distance BP ≈ The distance EP = r. Then, for small values of $\Delta\theta$, EB≈ $r\sin(\Delta\theta) \approx r\Delta\theta$.

Consider next the right angle triangle ABE. For this triangle, EB/AB = $\sin\theta$. Using

EB = $r\Delta\theta$, and AB = δ we get

$$\Delta q = d \sin q \, / \, r$$

Thus, stream function associated with the doublet, in the limit as δ goes to zero, is given by:

$$\boxed{y_{\,Doublet=} = -\frac{m}{2p}\frac{\sin q}{r}}$$

(27)

If we superpose the doublet, and a uniform flow we get:

$$y = u_\infty y - v_\infty x - \frac{m}{2p}\frac{\sin q}{r}$$

If the uniform flow is parallel to the x- axis, using $y = r\sin\theta$, and defining $a^2 = \mu / 2\pi V_\infty$ we get:

$$y = u_\infty y\left(1 - \frac{a^2}{r^2}\right)$$

(28)

Notice that the stream function ψ is zero on the surface r=a. In other words, r=a, a cylinder is a streamline. Thus, the superposition of a doublet and a uniform flow, for some special situations, becomes flow over a circular cylinder. We can plot this function using MATLAB. We will find that this function does yield flow over a circular cylinder of radius a.

Chapter III. Part III

Flow over a Circular Cylinder

In the previous handout we derived the following equation for the doublet:

$$\boxed{\psi_{Doublet=} = -\frac{\mu}{2\pi}\frac{\sin\theta}{r}} \tag{29}$$

The quantity μ is called the doublet strength. When this doublet is superposed over a uniform flow parallel to the x- axis, we get:

$$\psi = u_\infty y - \frac{\mu}{2\pi}\frac{\sin\theta}{r} \tag{30}$$

The above equation contains some terms in the Cartesian coordinate system, while the second term is in the polar coordinate system. To avoid unnecessary confusion, we express y as rsinθ . Then, equation (30) becomes:

$$\psi = u_\infty r\sin\theta - \frac{\mu}{2\pi}\frac{\sin\theta}{r} = u_\infty\left(1 - \frac{\mu}{2\pi u_\infty}\frac{1}{r^2}\right) r\sin\theta \tag{31}$$

If we replace the constant $\frac{\mu}{2\pi u_\infty}$ by a new constant R^2, the above equation becomes:

$$\boxed{\psi = u_\infty\left(1 - \frac{R^2}{r^2}\right) r\sin\theta} \tag{32}$$

Let us examine this equation. At all points where r=R (i.e. at all points on a circle of radius R), the stream function is zero. Secondly, taking the derivative of the above stream function we get the following expression for the radial velocity:

$$v_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} = u_\infty\left(1 - \frac{R^2}{r^2}\right)\cos\theta \tag{33}$$

This radial velocity is zero on all points on the circle r=R. That is, there can be no velocity normal to the circle r=R. Thus this circle itself is a streamline.

If we plot the stream function shown on (4) using MATLAB, you will find that the streamlines indeed flow around a circular cylinder of radius R. One of these streamlines that originally started along the x- axis will continue as a straight line until it reaches the front nose of the cylinder. There it will split into two parts, and flow over the upper and lower sides, respectively. It will reunite with its twin sibling at the rear of the cylinder, and proceed along the x- axis.

We can also compute the tangential component of velocity for flow over the circular cylinder. From equation (4),

$$v_q = -\frac{\partial y}{\partial r} = -u_\infty\left(1+\frac{R^2}{r^2}\right)\sin q \tag{34}$$

<u>Non-dimensional Form of Velocities and Pressure</u>: Notice that the tangential velocity given by equation (34) and the radial velocity given by equation (33) simply scale by the free-stream velocity u_∞ . Thus, we can work with the non-dimensional forms of velocity:

$$\frac{v_q}{u_\infty} = -\left(1+\frac{R^2}{r^2}\right)\sin q$$

$$\frac{v_r}{u_\infty} = \left(1-\frac{R^2}{r^2}\right)\cos q \tag{37}$$

Notice that the distances are non-dimensionalized by the cylinder radius R. We can solve the problem for a single radius R=1, and a single freestream velocity (equal to unity). We can scale up this solution to any cylinder radius (by simply multiplying the distance r by the actual cylinder radius R) and any freestream velocity by simply multiplying the non-dimensional velocity by the actual freestream velocity. This scaling is simple and linear. It is like enlarging a single negative to get small valet-size pictures or large pictures that go on a fire place!

On the surface of the cylinder r=R, we get the following expression for the tangential and radial components of velocity:

$$\boxed{v_q = -2u_\infty \sin q} \tag{35}$$

$$\boxed{v_r = 0} \tag{36}$$

How about pressure p? What is the best way to non-dimensionalize it? To answer this, we look at the Bernoulli's equation:

$$\frac{p}{r}+\frac{1}{2}\left(u^2+v^2\right)=\frac{p_\infty}{r}+\frac{1}{2}\left(u_\infty^2+v_\infty^2\right)=\frac{p_\infty}{r}+\frac{1}{2}V_\infty^2$$

After some rearrangement we get the following non-dimensional form:

$$C_p = \frac{p - p_\infty}{\frac{1}{2} r V_\infty^2} = 1 - \frac{u^2 + v^2}{V_\infty^2}$$

(37)

This non-dimensional form C_p, called the pressure coefficient, is the single quantity we need to compute. We can always recover the dimensional p for any freestream velocity V_∞ if we know C_p.

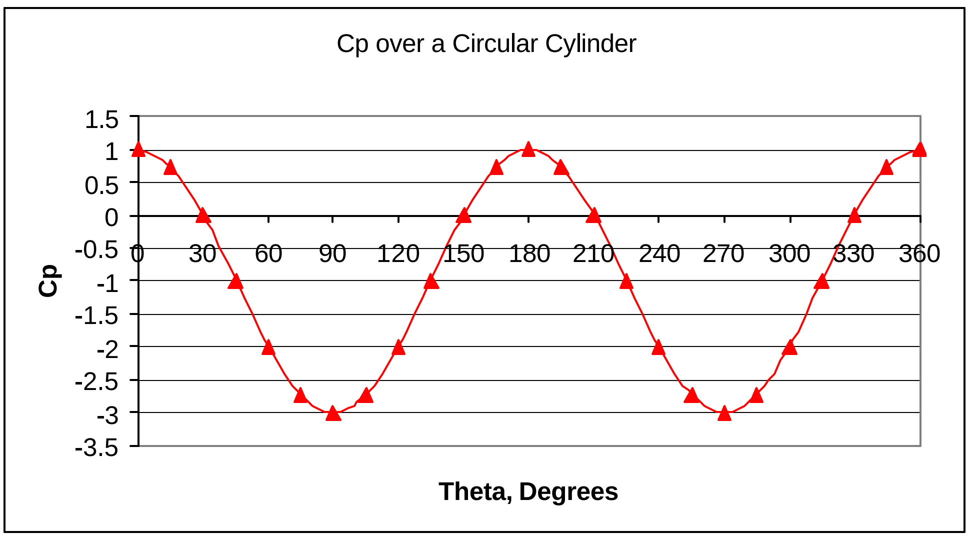
For the circular cylinder being studied here, at the surface, the only velocity component that is non-zero is the tangential component of velocity vq. Using (35) in (37), we get at the cylinder surface the following expression for the pressure coefficient:

$$C_p = 1 - 4\sin^2 q$$

(38)

where θ is the angle measured from the rear stagnation point (at the intersection of the back end of the cylinder with the x- axis).

Let us plot Cp against θ, in the range between 0 and 360 degrees.

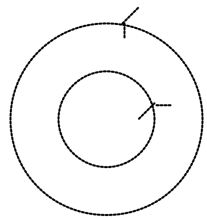


It is seen that it is symmetric about the y- axis (θ = 90 and θ = 270 degrees). Because Cp exhibits this fore and aft-symmetry there is no drag according to the potential flow theory. In real flows, the flow separates over the rear side of the cylinder creating a wake. Thus, the pressure in the front and the aft are not symmetric, and drag is produced. This inconsistency between the potential flow theory and reality was pointed out by dÁlembert, and is commonly called the dÁlembert Paradox.

The above pressure distribution is also symmetric about the x- axis. That is, the upper and lower sides of the cylinder have the same pressure distribution. As a consequence, there is no net pressure force pushing the cylinder up or down. There is no lift.

Building Block: Velocity Field and Stream Function Associated with a Point Vortex:

We next turn our attention to the next, and the last, building block of potential flow called the "Point Vortex". It may be thought of a finite amount of vorticity concentrated at a single point in space producing a concentric streamlines shown as below. The vorticity is stored at the center of the circles.



Consider one of the circular streamlines of radius r. Let v_θ be the tangential velocity at any point on this circle. Because of the inherent symmetry of this flow, v_θ will not change with θ. The circulation associated with this closed contour is:

$$Circlulation = \oint \vec{V}.d\vec{s} = 2\pi r v_\theta \tag{3}$$

According to Stokes'theorem, this circulation must equal all the vorticity contained in the area enclosed by this circular streamline. In our case, the vorticity is concentrated at just the origin. Let this vorticity be given the symbol Γ, called the strength of the vortex. Then,

$$\Gamma = 2\pi r v_\theta \tag{4}$$

or,

$$v_\theta = \frac{\Gamma}{2\pi r} \tag{5}$$

Using the fact that $v_\theta = -\frac{\partial \psi}{\partial r} = \frac{1}{r}\frac{\partial \phi}{\partial \theta}$, we get:

$$\psi = -\frac{\Gamma}{2\pi}\log r \tag{6}$$

Lifting Flow over a Circular Cylinder:

Let us now superpose the flow over a circular cylinder that we studied in the previous handout, and the vortex solution studied here, We assume that the vortex is placed at the origin (the center of the cylinder). When we superpose such basic building blocks, we can add or subtract a constant, because all the basic building blocks that we have derived to date are true to within a constant. Adding or subtracting the constant will not affect the ability of these basic building block solutions from satisfying the Laplace's equation. To the point vortex stream function we have shown above, we will add the constant $(\Gamma/2\pi)\log R$.

Thus, the stream function associated with the flow over a circular cylinder, with a point vortex of strength Γ placed at the cylinder center is:

$$\boxed{\psi = u_\infty\left(1-\frac{R^2}{r^2}\right) r\sin\theta - \frac{\Gamma}{2\pi}\log\frac{r}{R}} \tag{7}$$

We can make several observations about this stream function:

i) At r=R, i.e. on the cylinder surface, the stream function is zero.

ii) The radial velocity is given by:

$$v_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} = u_\infty\left(1-\frac{R^2}{r^2}\right) r\cos\theta = 0 \tag{8}$$

This quantity is also zero at r=R, meaning that the flow field is tangential to the cylinder.

In other words, the circular cylinder surface is a streamline for the flow with a vortex, as was the case for the non-lifting cylinder.

iii) We can determine the stagnation points on the cylinder surface by determining the radial and tangential components of velocity to zero. The radial component is zero all over the cylinder. Let us set the tangential component to zero.

$$v_\theta = -\left.\frac{\partial\psi}{\partial r}\right|_{r=R} = -\left.\left(u_\infty\left(1+\frac{R^2}{r^2}\right)\sin\theta - \frac{\Gamma}{2\pi r}\right)\right|_{r=R} = -2u_\infty\sin\theta + \frac{\Gamma}{2\pi R} \tag{9}$$

We find that v_θ is zero if

$$\sin\theta = \frac{\Gamma}{4\pi u_\infty R} \tag{10}$$

If the absolute value of the expression on the right side is less than unity, then two solutions for θ exist, and there will be two stagnation points on the cylinder. Note that Γ is positive for counterclockwise vortices and Γ is a negative number for clockwise vortices. (Anderson uses the opposite convention.)

See figure 3-27 in the text for a view of the streamlines around the cylinder for various values of Γ, and how the stagnation point locations change with Γ.

iv) Since we know the tangential component of velocity at any point on the cylinder (and the radial component of velocity is zero), we can find the pressure field over the surface of the cylinder from Bernoulli's equation:

$$\frac{p}{r}+\frac{v_r^2}{2}+\frac{v_q^2}{2}=\frac{p_\infty}{r}+\frac{u_\infty^2}{2}$$

(11)

Using (8) and (9) in (11) we get:

$$p = p_\infty + \frac{1}{2} r\left[u_\infty^2 - \left\{2u_\infty \sin q - \frac{\Gamma}{2pR}\right\}^2\right] = \left(p_\infty + \frac{1}{2} r u_\infty^2 - \frac{r\Gamma^2}{2p^2 R^2}\right) - 2 r u_\infty^2 \sin^2 q + \frac{r u_\infty \Gamma}{pR}\sin q$$

$$= A + B\sin q + C\sin^2 q$$

where,

$$A = \left(p_\infty + \frac{1}{2} r u_\infty^2 - \frac{r\Gamma^2}{2p^2 R^2}\right)$$

$$B = \frac{r u_\infty \Gamma}{pR}$$

$$C = -2 r u_\infty^2$$

(12)

Calculation of Lift: Let us first consider lift. Lift per unit span, L (i.e. per unit distance normal to the plane of the paper) is given by:

$$L = \int_{Lower} pdx - \int_{upper} pdx$$

(13)

On the surface of the cylinder, x = Rcosθ. Thus, dx = -Rsinθdθ, and the above integrals may be thought of as integrals with respect to θ. For the lower surface, θ varies between π and 2π. For the upper surface, θ varies between π and 0. Thus,

$$L = -R\int_p^{2p}\left(A + B\sin q + C\sin^2 q\right)\sin q dq + R\int_p^0\left(A + B\sin q + C\sin^2 q\right)\sin q dq$$

(13)

Reversing the upper and lower limits of the second integral, we get:

$$L = -R\int_0^{2p}\left(A + B\sin q + C\sin^2 q\right)\sin q dq = -R\int_0^{2p}\left(A\sin q + B\sin^2 q + C\sin^3 q\right)dq = -BRp$$

(14)

Substituting for B from equation (12), we get:

$$\boxed{L = -r u_\infty \Gamma}$$

(15)

This is an important result. It says that clockwise vortices (negative numerical values of Γ) will produce positive lift that is proportional to Γ and the freestream speed. Kutta and Joukowski generalized this result to lifting flow over airfoils. Equation (15) is known as the Kutta-Joukowski theorem.

Anderson assumes that clockwise vortices are positive, while we have assumed that counterclockwise vortices are positive. This explains the difference of a negative sign in equation for lift between us and Anderson.

Drag: We can likewise integrate drag forces. The drag per unit span, D, is given by:

$$D = \int_{Front} p dy - \int_{rear} p dy$$

Since y=Rsinθ on the cylinder, dy=Rcosθdθ. Thus, as in the case of lift, we can convert these two integrals over y into integrals over θ. On the front side, θ varies from 3π/2 to π/2. On the rear side, θ varies between 3π/2 and π/2. Performing the integration, we can show that

$$\boxed{D = 0} \qquad (16)$$

This result is in contrast to reality, where drag is high due to viscous separation. This contrast between potential flow theory and drag is the dÁlembert Paradox.

Exercise: Complete the integration for drag, and verify that D is indeed zero.