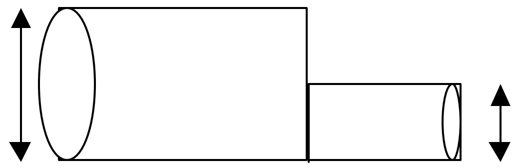
Chapter II Worked-Out Examples



5 feet dia 1 foot dia

1. Water is flowing through a large pipe of diameter 5 feet from left to right as shown. The velocity at the inlet is given by:

$$V = 6.25 - r^2 \text{ ft/sec}$$

What is the average velocity of the flow leaving through the smaller pipe of diameter 1 feet?

This is a steady flow. Continuity equation simplifies to $\oiint_{Inlet+Outlet} r\left[\vec{V} \cdot n\right] dS = 0$

At the inlet, the unit normal vector is $-\vec{i}$. At the outlet the unit normal vector is $\vec{i}$.

At the inlet, $\vec{V} = \left(6.25 - r^2\right)\vec{i}$ At the exit, the velocity is $V_{exit}\vec{i}$

We can assume that the inlet surface is made of annular strips of radius r, and width dr. Thus, dS=2πrdr

We get:

$$\oiint_{Inlet} r\left[\vec{V} \cdot n\right] dS == -r \int_0^{2.5\,feet} \left(6.25 - r^2\right) 2prdr = -2rp\left[6.25 * \frac{2.5^2}{2} - \frac{2.5^4}{4}\right] slugs/\sec$$

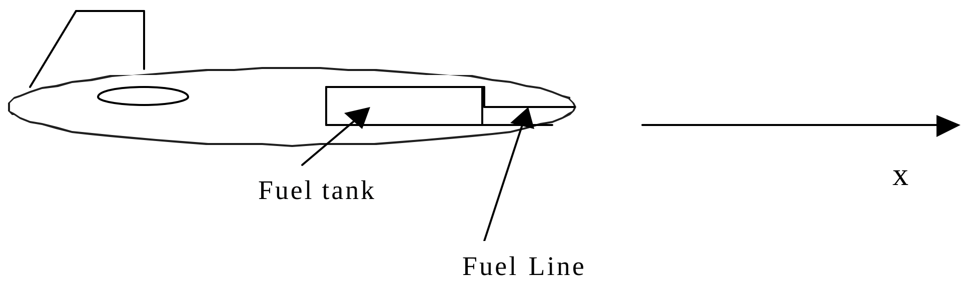
At the outlet, we get:

$$\oiint_{Outlet} r\vec{V} \cdot \vec{n} dS = rV_{Exit}A_{exit} = rV_{Exit}p\,(0.5)^2\, slugs/\sec$$

Sum up the integral at the inlet and at the outlet, and set the sum to zero as required by continuity.

$$-2rp\left[6.25 * \frac{2.5^2}{2} - \frac{2.5^4}{4}\right] slugs/\sec + rV_{Exit}p\,(0.5)^2\, slugs/\sec = 0$$

Solve for V_{exit}. After minor arithmetic, the exit velocity is 78.125 ft/sec



Exercise: A fighter aircraft is being refueled in flight at the rate of 150 gallons/minute. The specific gravity of the fuel is 0.68. The inside diameter of the fuel line is 5 inches. The fluid pressure at the entrance to the aircraft is 4 psi. What additional thrust does the plane need to develop to keep the aircraft at constant velocity during the refueling process?

Note: Density of water = 1.938 slug/ft^3.
1 gallon/minute = 0.002228 ft^3/sec

Solution:

Step1:We choose a coordinate system first. In this case, we choose the x-axis to be along the thrust direction, as shown above.

Step 2: The next step is to choose a control volume. The control volume is the fuel tank, and the fuel line or duct that leads from the fuel tank to the aircraft nose. We need not know the precise shape of the tank. Since my software has trouble drawing anything but rectangles and ellipses, I have drawn the tank and the fuel line as two rectangle-like objects.

Step 3: Identify what we are after. We are interested in computing the x- component of the normal force exerted by the fuel inside the control volume on the tank and fuel line walls. This normal force is pressure difference between the interior and exterior of the control volume walls(the exterior is vented to atmosphere) times the area. At each point on the tank/fuel line wall, the force will act along the unit normal vector $\vec{n}$. The total force along the x-direction is given by

$$\oiint_{\substack{\text{Fuel Line}\\ \text{and tank}}} (\mathrm{p} - \mathrm{p}_\infty)\vec{i} \bullet \vec{n} dS - F_{x,viscous} \qquad (1)$$

Recall that $F_{x,viscous}$ is the viscous force exerted by the surface S of the tank and pipes on the control volume. To every action, there is a reaction. Thus $-F_{x,viscous}$ is the viscous force exerted by the fluid on the walls, and the aircraft.

Step 4: From the integral form of momentum equation isolate what we are after. From the integral form of the u-momentum equation we know that:

$$\oiint (p - p_\infty)\ \vec{i} \bullet \vec{n} dS + \oiint r u \vec{V} \bullet \vec{n} dS = F_{x,viscous}$$

(2)

Here the integral is over all the boundaries of the control volume – the fuel tank, the fuel line and the entrance to the fuel line. We know that no fuel or momentum can escape through the walls of the fuel tank or line. Thus, from the above equation we can write:

$$\oiint_{\substack{\text{Fuel Tank}\\ \text{and}\\ \text{line}}} (p - p_\infty)\ \vec{i} \bullet \vec{n} dS - F_{x,viscous} = - \oiint_{\text{Entrance}} (p - p_\infty) \vec{i} \bullet \vec{n} dS - \oiint_{\textit{Entrance}} r u \vec{V} \bullet \vec{n} dS$$

(3)

As we saw earlier in expression (1), the left side of equation (3) represents the force exerted by the fluid inside the control volume on the fuel tank/fuel line and eventually on the aircraft.

Step 5: Evaluate! We can evaluate the right hand side of (3) as follows.

Volume rate of flow of the fuel	= 150 gallons/minute
	= 150 * 0.002228 ft^3/sec
	= 0.3342 ft^3/sec
Entrance radius	= 5/2 inches
	= 5/24 feet
Entrance area	= π * (5/24) * (5/24) ft^2
	= 0.1364 ft^2
Velocity at the entrance	= Volume rate of flow/Entrance area
	= 0.3342/0.1364 ft/sec
	= 2.45 ft/sec

This x- component of velocity, given the symbol u, will be along the negative x- axis. Therefore it is given a sign as well. Thus,

$$\vec{V} = -2.45\vec{i}$$

u = - 2.45 ft/sec

At the entrance, the unit normal $\vec{n}$ is along the +x direction. It therefore equals + $\vec{i}$.

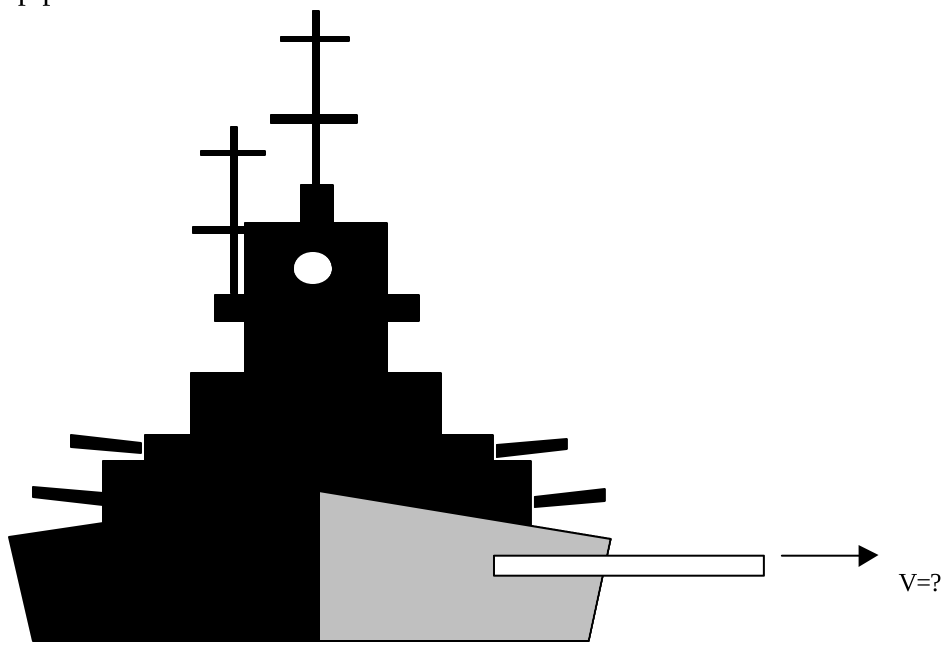
At the entrance,	$p - p_\infty$	= 4 psi
		= 576 lbf/ft^2
At the entrance,	$\vec{V} \bullet \vec{n}$	= - 2.45 ft/sec
Density of fuel		= 0.68 * 1.938 slug/ft^3
		= 1.31784 slug/ ft^3

The right hand side thus becomes: - {576 * 0.1364}
- {1.31784 * (-2.45) (-2.45) * 0.1364} lbf.
= - 78.5664 – 1.08

Performing the arithmetic, the right hand side becomes – 79.64 lbf.

This is the force that the fluid inside the tank will exert on the tanks, and eventually on the aircraft. The negative sign tells us that it is directed along the –ve x- axis pushing the aircraft back. Thus 79.64 lbf of extra thrust is needed to overcome it.

Problem 3: Find the velocity that a water jet will use to propel a ship with a thrust force equal to 3000 lb. The pipe diameter is 2 ft.



Solution: This problem is similar to the aircraft-refueling problem we studied before. In that problem, fuel was entering the aircraft. Here the water jet is leaving the ship through a pipe of diameter 2 ft.

Step 1: We need to choose the x-axis. We will assume that it is pointing to the right hand side. Then, at the outlet, the normal vector is along the x- axis, and equals i .
Step 3: As in the aircraft-refueling problem, we will construct a control volume – this will be a water tank and pipe of unknown shape where the water is stored. The only boundary through which water can enter or leave is the pipe exit.

Step 3: Identify what we are after. Now, the force on the ship is due to pressure exerted by the fluid on the tank and pipe walls and the viscous forces. This is given by:

$$\boxed{\text{Force on the ship } = \oiint_{Tank+Pipe} (p - p_{atm})\vec{n} \cdot \vec{i}\, dS - F_{x,viscous}}$$

Step 4: From the momentum equation, isolate the terms we need.

The steady state form of the momentum equation states:

$$\boxed{\oiint_S p\vec{n} \cdot \vec{i}\, dS + \oiint_S ru\vec{V} \cdot \vec{n} dS = F_{x,viscous}} \tag{1}$$

Here S is the boundary that includes the tank walls, the pipe wall and the pipe exit (outlet).

We can subtract atmospheric from the pressure p in the above equation. As we discussed in the class, adding or subtracting a constant from the pressure field does not change the pressure force acting on a closed volume. Thus, the above equation may be viewed as:

$$\oiint_{Tank+Pipe} (p - p_{atm})\vec{n} \cdot \vec{i}\, dS + \oiint_{Tank+Pipe} ru\vec{V} \cdot \vec{n} dS + \oiint_{Outlet} (p - p_{atm})\vec{n} \cdot \vec{i}\, dS + \oiint_{Outlet} ru\vec{V} \cdot \vec{n} dS = F_{x,viscous} \quad (2)$$

At the pipe and tank walls, the normal component of velocity is zero. Thus, the second term in equation (2) is zero. The third term in equation (2) is also zero since at the pipe outlet the pressure p equals atmospheric pressure (given). Thus, the force on the ship reduces to:

$$\text{Force on the ship} = \oiint_{Tank+Pipe} (p - p_{atm})\vec{n} \cdot \vec{i}\, dS - F_{x,viscous} = -\oiint_{outlet} ru\vec{V} \cdot \vec{n} dS$$

Step 5: Evaluate the quantities. Assuming uniform flow at the outlet, the right hand side is $-\rho V^2 A$ where A is the pipe cross section area. Thus,

$$\text{Force on the ship} = -\rho V^2 A$$

The negative sign simply indicates that this thrust will act along the negative direction, i.e. to the left, pushing the ship towards the left.

We are given the magnitude of F as 3000 lbf. Substitute for density $\rho = 1.938$, use $A = \pi R^2$

We can solve for V, giving $V = \sqrt{\dfrac{3000}{1.938 \cdot p \cdot 1}} = 22.2\, ft/s$